Corrigendum: SE-Sync: A Certifiably Correct Algorithm for Synchronization over the Special Euclidean Group

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Abstract

This short note presents a correction for an error in the proof of Theorem 12 in the article "SE-Sync: A Certifiably Correct Algorithm for Synchronization over the Special Euclidean Group" [3] and its companion technical report [2]. Theorem 12 itself remains correct as originally stated.

1 Context

The article [3] and its companion technical report [2] describe an efficient algorithmic approach for computing solutions of the following *special Euclidean synchronization problem* in a non-adversarial noise regime: estimate the values of a set of n unknown group elements $\underline{x}_1, \dots, \underline{x}_n \in \mathrm{SE}(d)$ given noisy measurements $\tilde{x}_{ij} \in \mathrm{SE}(d)$ of a subset $\vec{\mathcal{E}} \subset [n] \times [n]$ of their pairwise relative transforms $\underline{x}_{ij} \triangleq \underline{x}_i^{-1}\underline{x}_j$. Under a suitable noise model, this problem can be formalized as the following maximum-likelihood estimation [3, Sec. 3.2]:

$$p^*_{\mathrm{MLE}} = \min_{\substack{t_i \in \mathbb{R}^d \\ R_i \in \mathrm{SO}(d)}} \sum_{(i,j) \in \vec{\mathcal{E}}} \kappa_{ij} \|R_j - R_i \tilde{R}_{ij}\|_F^2 + \tau_{ij} \left\| t_j - t_i - R_i \tilde{t}_{ij} \right\|_2^2, \tag{1}$$

where we have exploited the identification $\mathrm{SE}(d) \cong \mathbb{R}^d \rtimes \mathrm{SO}(d)$ in order to write an element $x \in \mathrm{SE}(d)$ componentwise as $x = (t, R)$ for $t \in \mathbb{R}^d$ and $R \in \mathrm{SO}(d)$.

After some algebraic manipulation (and a relaxation from $\mathrm{SO}(d)$ to $\mathrm{O}(d)$), one can derive the following estimator for the orientation components of the unknown states [3, Sec. 4]:

$$p^*_{\mathrm{O}} = \min_{R \in \mathrm{O}(d)^n} \operatorname{tr}(\tilde{Q} R^\mathsf{T} R); \tag{2}$$

here $R \triangleq (R_1, \dots, R_n)$ is a $d \times dn$ block matrix obtained by concatenating the orientations $R_1, \dots, R_n \in \mathrm{O}(d)$, and $\tilde{Q} \in \mathbb{S}^{dn}_+$ is a symmetric data matrix constructed from the measurements $\tilde{x}_{ij}$. Note that the objective in (2) is invariant under the diagonal left-action $\bullet$ of $\mathrm{O}(d)$ on $\mathrm{O}(d)^n$:

$$G \bullet (R_1, \dots, R_n) \triangleq (GR_1, \dots, GR_n), \tag{3}$$

and therefore minimizers of (2) are *non-unique*; indeed, given any $R \in \mathrm{O}(d)^n$, the orbit:

$$\mathcal{O}(R) \triangleq \{G \bullet R \mid G \in \mathrm{O}(d)\} \subset \mathrm{O}(d)^n \tag{4}$$

generated by R under $\bullet$ is comprised of "equivalent" candidate solutions of (2). In consequence, when quantifying the estimation error of an estimate R^* of $\underline{R}$ obtained as a minimizer of (2), it is important to do so in a manner that accounts for this gauge symmetry. To that end, [2, 3] make use of the following *orbit distance*:

$$d_{\mathcal{O}}(X, Y) \triangleq \min_{G \in \mathrm{O}(d)} \|X - G \bullet Y\|_F, \tag{5}$$

which measures the Frobenius-norm distance between the two nearest representatives of the orbits $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ determined by X and Y, respectively. Theorem 12 of [2, 3] then provides an upper bound on the estimation error $d_{\mathcal{O}}(\underaccent{\tilde}{R}, R^*)$, which we restate here for convenience:

Theorem (Theorem 12 of [2, 3])**.** *Let $\underaccent{\tilde}{Q}$ be the data matrix of the form appearing in* (2) *constructed using the true relative transforms $\underaccent{\tilde}{x}_{ij} = (\underaccent{\tilde}{t}_{ij}, \underaccent{\tilde}{R}_{ij})$, $\underaccent{\tilde}{R} \in \mathrm{SO}(d)^n$ the matrix composed of the true rotational states, and $R^* \in \mathrm{O}(d)^n$ an estimate of $\underaccent{\tilde}{R}$ obtained as a minimizer of* (2)*. Then the estimation error $d_{\mathcal{O}}(\underaccent{\tilde}{R}, R^*)$ admits the following upper bound:*

$$\sqrt{\frac{4dn\|\tilde{Q} - \underaccent{\tilde}{Q}\|_2}{\lambda_{d+1}(\underaccent{\tilde}{Q})}} \geq d_{\mathcal{O}}(\underaccent{\tilde}{R}, R^*). \tag{6}$$

2 Error in the proof of Theorem 12

The error in the proof of Theorem 12 occurs in the argument used to establish the following inequality (cf. equations (135)–(138) of [3], corresponding to equations (147)–(150) of [2]):

$$\|P\|_F^2 \geq \frac{1}{2} d_{\mathcal{O}}(\underaccent{\tilde}{R}, R^*)^2, \tag{7}$$

where P is the orthogonal projection of R^* onto the orthogonal complement of $\underaccent{\tilde}{R}$ in $\mathrm{O}(d)^n$ under the Frobenius inner product.

Theorem 5 of [2, 3] establishes:

$$d_{\mathcal{O}}(\underaccent{\tilde}{R}, R^*)^2 = 2dn - 2\left\|\underaccent{\tilde}{R}R^{*\mathsf{T}}\right\|_* \tag{8}$$

and [3, eq. (131)] (resp. [2, eq. (143)]) derives:

$$\|P\|_F^2 = dn - \frac{1}{n}\left\|\underaccent{\tilde}{R}R^{*\mathsf{T}}\right\|_F^2. \tag{9}$$

Letting

$$\underaccent{\tilde}{R}R^{*\mathsf{T}} = U \operatorname{Diag}(\sigma_1, \ldots, \sigma_d) V^{\mathsf{T}} \tag{10}$$

be a singular value decomposition of $\underaccent{\tilde}{R}R^{*\mathsf{T}}$, equation (8) is equivalent to:

$$d_{\mathcal{O}}(\underaccent{\tilde}{R}, R^*)^2 = 2dn - 2\sum_{i=1}^{d} \sigma_i, \tag{11}$$

and

$$\left\|\underaccent{\tilde}{R}R^{*\mathsf{T}}\right\|_F^2 = \sum_{i=1}^{d} \sigma_i^2. \tag{12}$$

In light of (9), (11), and (12), the argument for (7) in [2, 3] attempts to establish an *upper bound* ϵ^2 for $\|\underaccent{\tilde}{R}R^{*\mathsf{T}}\|_F^2$ in terms of $\delta^2 \triangleq d_{\mathcal{O}}(\underaccent{\tilde}{R}, R^*)^2$ as the optimal value of the following constrained optimization problem:

$$\begin{aligned} \epsilon^2 = \max_{\sigma_i \geq 0} \sum_{i=1}^{d} \sigma_i^2 \\ \text{s.t.} \quad 2dn - 2\sum_{i=1}^{d} \sigma_i = \delta^2 \end{aligned} \tag{13}$$

(cf. equations (135) of [3] and (147) of [2]); note that here the decision variables σ_i are each constrained to be nonnegative, since they represent singular values.

The argument proceeds by introducing a Lagrange multiplier $\lambda \in \mathbb{R}$ corresponding to a first-order KKT point $\sigma^* \in \mathbb{R}^d$ of (13) satisfying:

$$\begin{aligned} 2\sigma^* + 2\lambda \mathbb{1}_d = 0, \\ 2dn - 2\sum_{i=1}^{d} \sigma_i^* = \delta^2. \end{aligned} \tag{14}$$

Solving (14) produces:

$$\sigma^* = \left(n - \frac{\delta^2}{2d}\right) \mathbb{1}_d, \tag{15}$$

corresponding to an objective value in (13) of:

$$\sum_{i=1}^{d} (\sigma_i^*)^2 = d\left(n - \frac{\delta^2}{2d}\right)^2. \tag{16}$$

Unfortunately, while σ^* is indeed a KKT point of (13), it is *not* a maximizer. As observed by Preskitt [1, Appendix D], equation (13) is equivalent to:

$$\begin{aligned} \epsilon^2 = \max_{\sigma \in \mathbb{R}^d} \|\sigma\|_2^2 \\ \text{s.t.} \quad \|\sigma\|_1 = dn - \frac{\delta^2}{2}. \end{aligned} \tag{17}$$

Standard norm inequalities give:

$$\frac{1}{\sqrt{d}} \|x\|_1 \leq \|x\|_2 \leq \|x\|_1, \tag{18}$$

with equality on the left for $x = c\mathbb{1}_d$ and equality on the right for $x = ce_i$, where $c \in \mathbb{R}$. In particular, we see that the KKT point σ^* identified in (15) is actually a *minimizer* of (13), rather than a *maximizer*. The error in the derivation of (15) is that the KKT system (14) does not include the inequality constraints present in the original problem (13), and in fact the maximizers

$$\mu_i^* \triangleq \left(dn - \frac{\delta^2}{2}\right) e_i, \quad i \in [d] \tag{19}$$

of (13) occur at points where at least one of the inequality constraints is binding.

In light of (17) and (18), the correct optimal value of (13) is:

$$\epsilon^2 = \left(dn - \frac{\delta^2}{2}\right)^2 = d^2\left(n - \frac{\delta^2}{2d}\right)^2, \tag{20}$$

which is greater than (16) by a factor of d. Unfortunately, this corrected bound is no longer sufficient to establish (7) (and, by extension, Theorem 12) using the original argument of [2, 3]; indeed, propagating the correction (20) through equations (139)–(140) of [3] (resp. (151)–(152) of [2]) produces:

$$\|P\|_F^2 \geq \frac{d}{2} d_{\mathcal{O}}(\underline{R}, R^*)^2 - dn(d-1), \tag{21}$$

which is weaker than the trivial (and asymptotically sharp) bound $\|P\|_F^2 \geq 0$ by a large constant $dn(d-1)$ in the limit $d_{\mathcal{O}}(\underline{R}, R^*) \to 0^+$.

3 An alternative argument

Fortunately, it is still possible to establish (7) using an alternative (and more elegant) argument, also due to Preskitt [1, Appendix D]. In light of (8) and (9), inequality (7) is equivalent to:

$$\frac{1}{n} \left\| \underline{R} R^{*\mathsf{T}} \right\|_F^2 \leq \left\| \underline{R} R^{*\mathsf{T}} \right\|_* . \tag{22}$$

Once again making use of the singular value decomposition (10), and defining $\sigma \triangleq (\sigma_1, \dots, \sigma_d)$, (22) is in turn equivalent to:

$$\frac{1}{n} \|\sigma\|_2^2 \leq \|\sigma\|_1. \tag{23}$$

It therefore suffices to establish (23). By Hölder's inequality:

$$\|\sigma\|_2^2 \leq \|\sigma\|_1 \|\sigma\|_\infty, \tag{24}$$

and

$$\|\sigma\|_\infty = \left\| \underline{R} R^{*\mathsf{T}} \right\|_2 = \left\| \sum_{i=1}^n \underline{R}_i R_i^{*\mathsf{T}} \right\|_2 \leq \sum_{i=1}^n \left\| \underline{R}_i R_i^{*\mathsf{T}} \right\|_2 = n \tag{25}$$

since $\underline{R}_i R_i^{*\mathsf{T}}$ is orthogonal. Substituting (25) into (24) and dividing by n produces (23). We therefore conclude that (7) holds, as desired.

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References

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