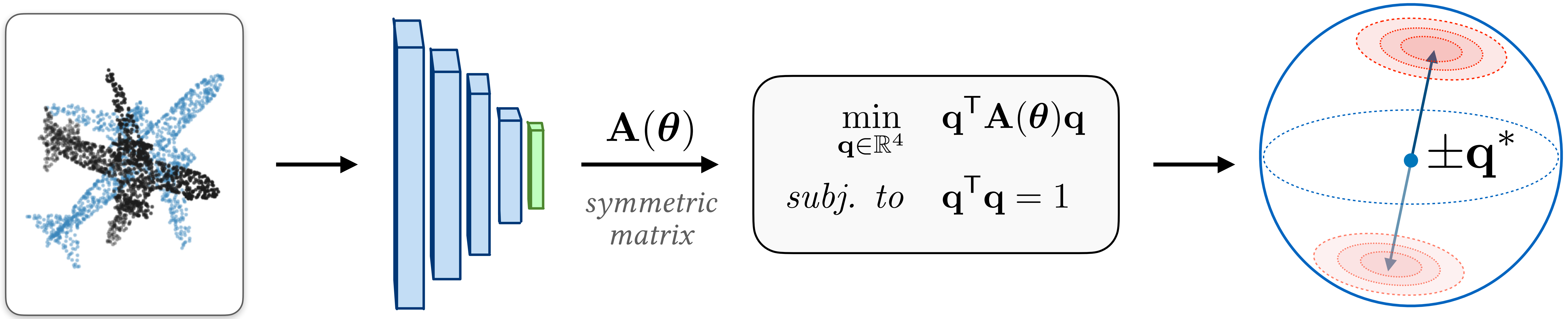
A Smooth Representation of Belief over SO(3)

for Deep Rotation Learning with Uncertainty

Robotics: Science and Systems 2020



Valentin Peretroukhin, Matthew Giamou, David M. Rosen, W. Nicholas Greene, Nicholas Roy, and Jonathan Kelly

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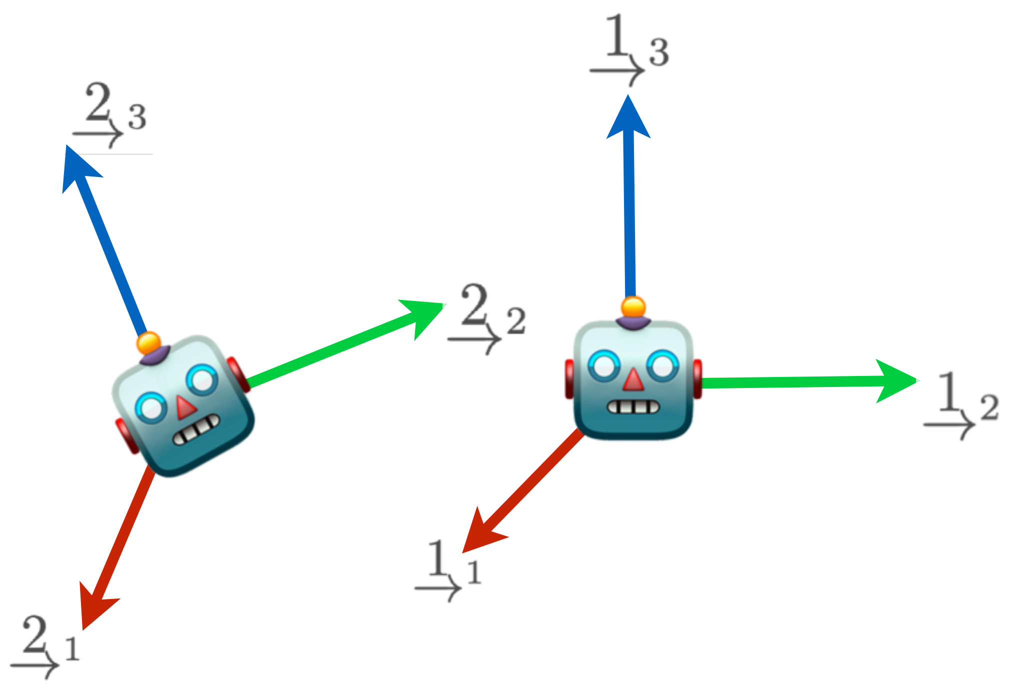
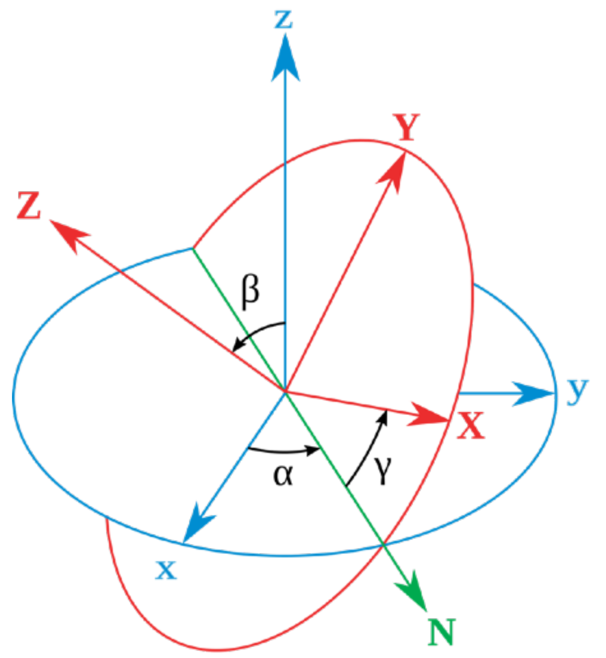
3D Rotation Group, SO(3)



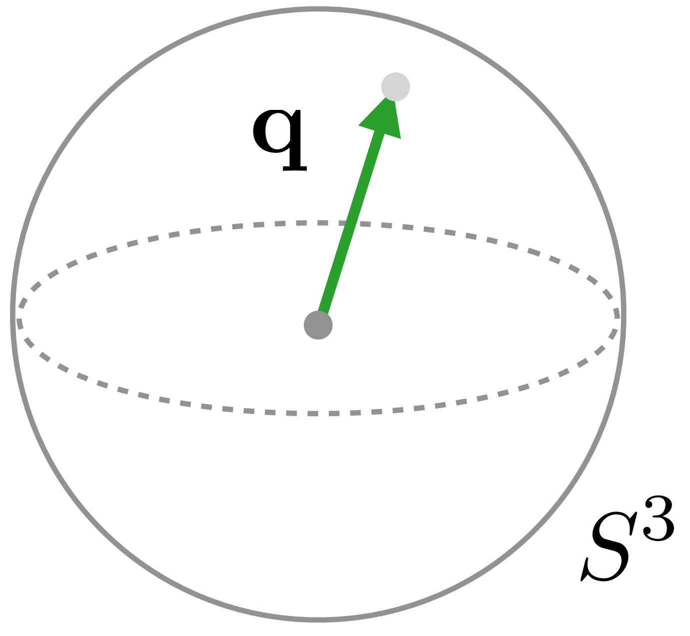
$$\begin{bmatrix} \underset{\rightarrow}{2}_1\cdot\underset{\rightarrow}{1}_1 & \underset{\rightarrow}{2}_1\cdot\underset{\rightarrow}{1}_2 & \underset{\rightarrow}{2}_1\cdot\underset{\rightarrow}{1}_3 \\ \underset{\rightarrow}{2}_2\cdot\underset{\rightarrow}{1}_1 & \underset{\rightarrow}{2}_2\cdot\underset{\rightarrow}{1}_2 & \underset{\rightarrow}{2}_2\cdot\underset{\rightarrow}{1}_3 \\ \underset{\rightarrow}{2}_3\cdot\underset{\rightarrow}{1}_1 & \underset{\rightarrow}{2}_3\cdot\underset{\rightarrow}{1}_2 & \underset{\rightarrow}{2}_3\cdot\underset{\rightarrow}{1}_3 \end{bmatrix}$$

Rotation matrices



$\{\alpha, \beta, \gamma\}$

Euler angles



Unit quaternions

$$\begin{bmatrix} | & & | \\ b_1 & \dots & b_n \\ | & & | \end{bmatrix}$$

$$b_3 = b_1 \times b_2$$

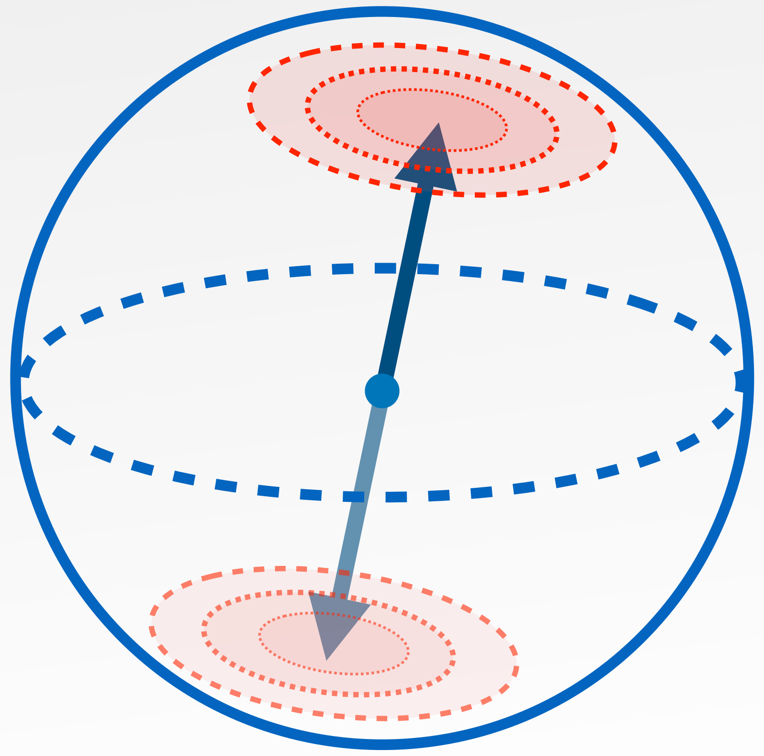
6D *continuous* representation

Zhou et al. CVPR (2019)

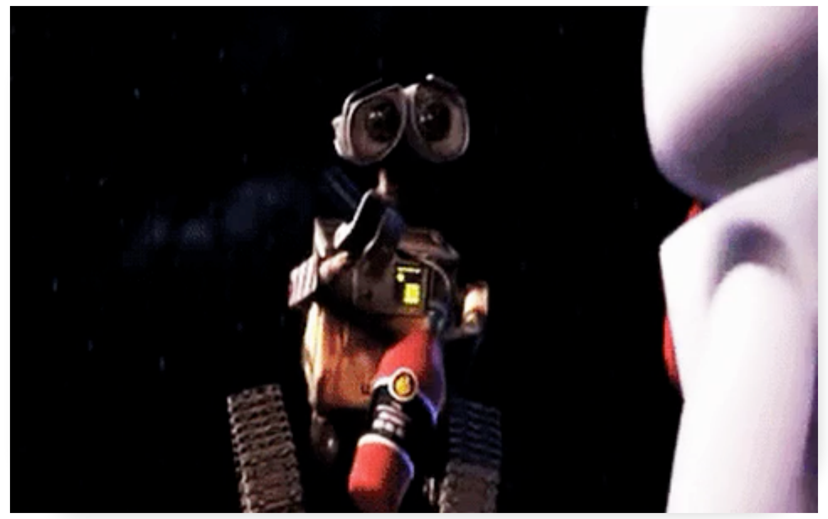
$\mathbf{A}(\boldsymbol{\theta})$

4x4 symmetric matrix

our *smooth* representation



The *Parametric* Wahba Problem



Problem 65–1, A Least Squares Estimate of Satellite Attitude, by Grace Wahba (International Business Machines Corporation).

Given two sets of n points $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$, and $\{\mathbf{v}_1^*, \mathbf{v}_2^*, \cdots, \mathbf{v}_n^*\}$, where $n \geqq 2$, find the rotation matrix M (i.e., the orthogonal matrix with determinant $+1$) which brings the first set into the best least squares coincidence with the second. That is, find M which minimizes

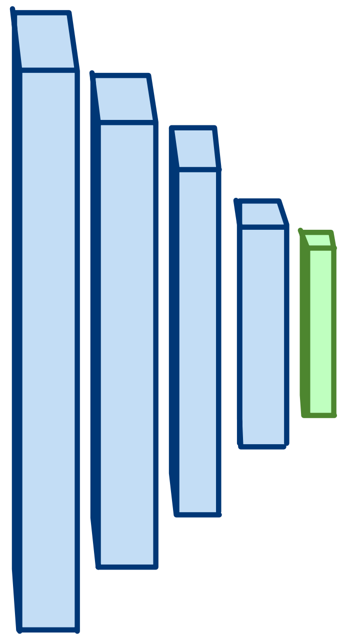
$$\sum_{j=1}^{n} \| \mathbf{v}_j^* - M\mathbf{v}_j \|^2.$$

Grace Wahba, SIAM Review (1965)

convert to ***QCQP*** *with data matrix* ***A***

$$\min_{\mathbf{q}\in\mathbb{R}^4} \quad \mathbf{q}^{\mathsf{T}}\mathbf{A}\mathbf{q} \qquad \textit{subj. to} \quad \mathbf{q}^{\mathsf{T}}\mathbf{q} = 1,$$

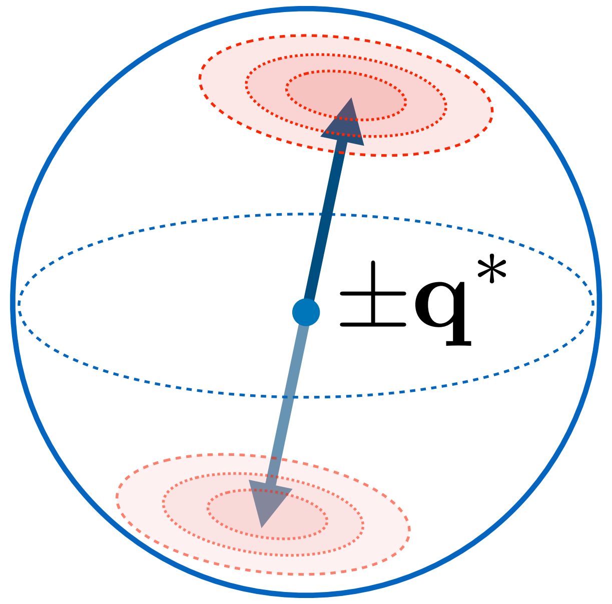
generalize to learned ***A***



$\mathbf{A}(\boldsymbol{\theta})$

$$\min_{\mathbf{q}\in\mathbb{R}^4} \quad \mathbf{q}^{\mathsf{T}}\mathbf{A}(\boldsymbol{\theta})\mathbf{q} \qquad \textit{subj. to} \quad \mathbf{q}^{\mathsf{T}}\mathbf{q} = 1$$

Parametric *QCQP*



Mapping $\mathbf{A}$ to SO(3) via a Differentiable Layer

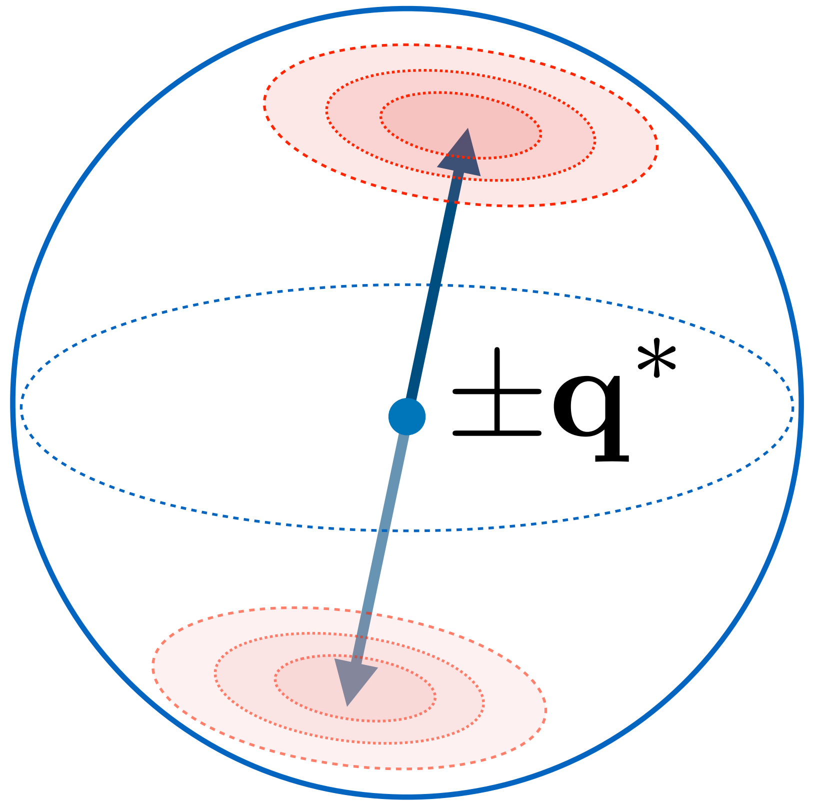
$$\mathbf{A}(\boldsymbol{\theta})$$

$$\begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \\ & \theta_5 & \theta_6 & \theta_7 \\ & & \theta_8 & \theta_9 \\ & & & \theta_{10} \end{bmatrix} \longrightarrow$$

minimum-eigenspace

$$\begin{aligned} \min_{\mathbf{q}\in\mathbb{R}^4} \quad & \mathbf{q}^\intercal \mathbf{A}(\boldsymbol{\theta})\mathbf{q} \\ \textit{subj. to} \quad & \mathbf{q}^\intercal\mathbf{q} = 1 \end{aligned} \longrightarrow$$

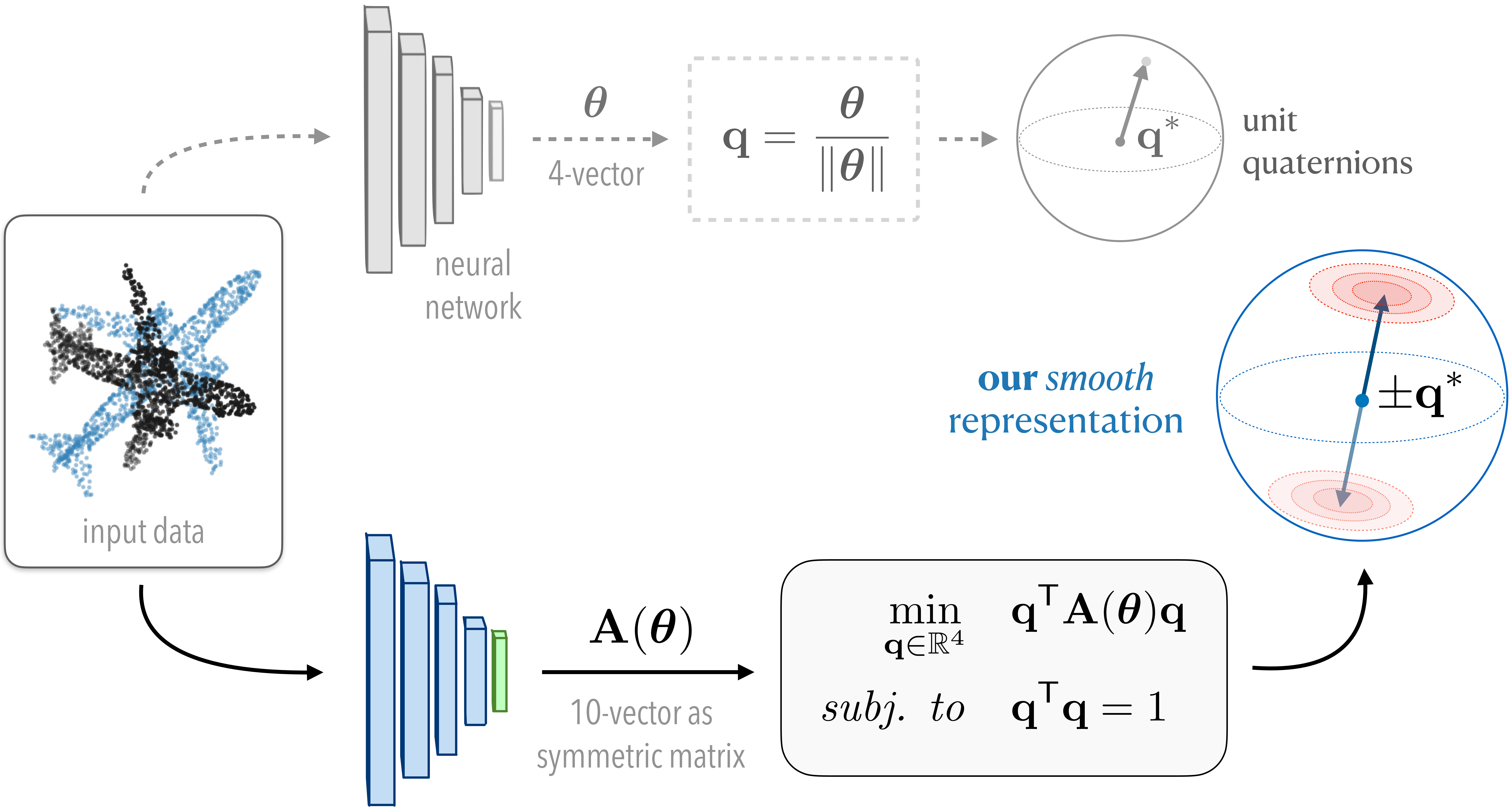
implicit function theorem



Jan Magnus, *'On Differentiating Eigenvalues and Eigenvectors…'*
Econometric Theory (1985)

Very simple in **PyTorch**!

```
_, evs = torch.symeig(A, eigenvectors=True)
```



input data
neural
network
$\boldsymbol{\theta}$
4-vector
$\mathbf{q} = \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|}$
$\mathbf{q}^*$
unit
quaternions
our *smooth*
representation
$\pm\mathbf{q}^*$
$\mathbf{A}(\boldsymbol{\theta})$
10-vector as
symmetric matrix
$\min_{\mathbf{q}\in\mathbb{R}^4} \quad \mathbf{q}^\intercal\mathbf{A}(\boldsymbol{\theta})\mathbf{q}$
$\textit{subj. to} \quad \mathbf{q}^\intercal\mathbf{q} = 1$

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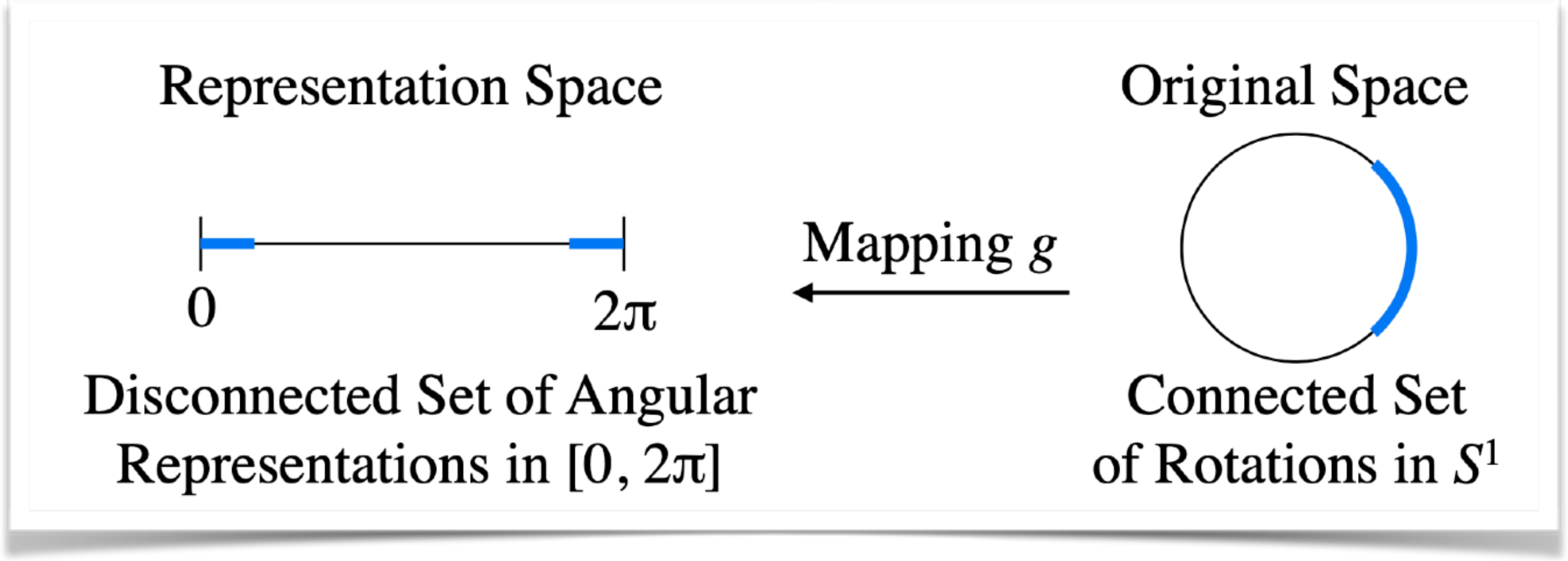
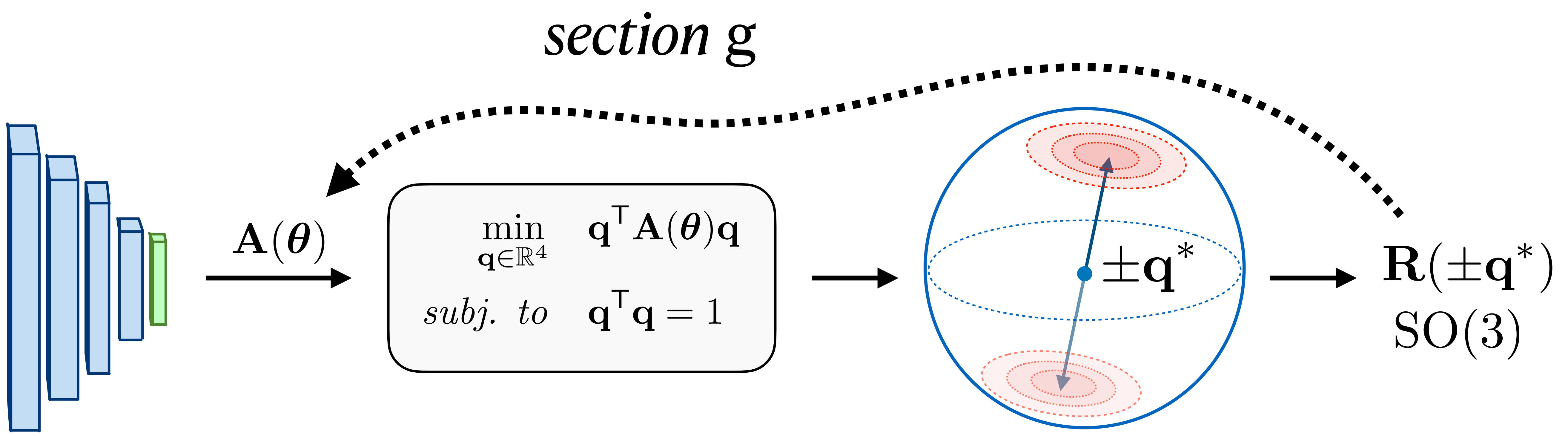
Continuity of SO(3) Representations



Zhou et al. *'On the Continuity of Rotation Representations…'* CVPR (2019)

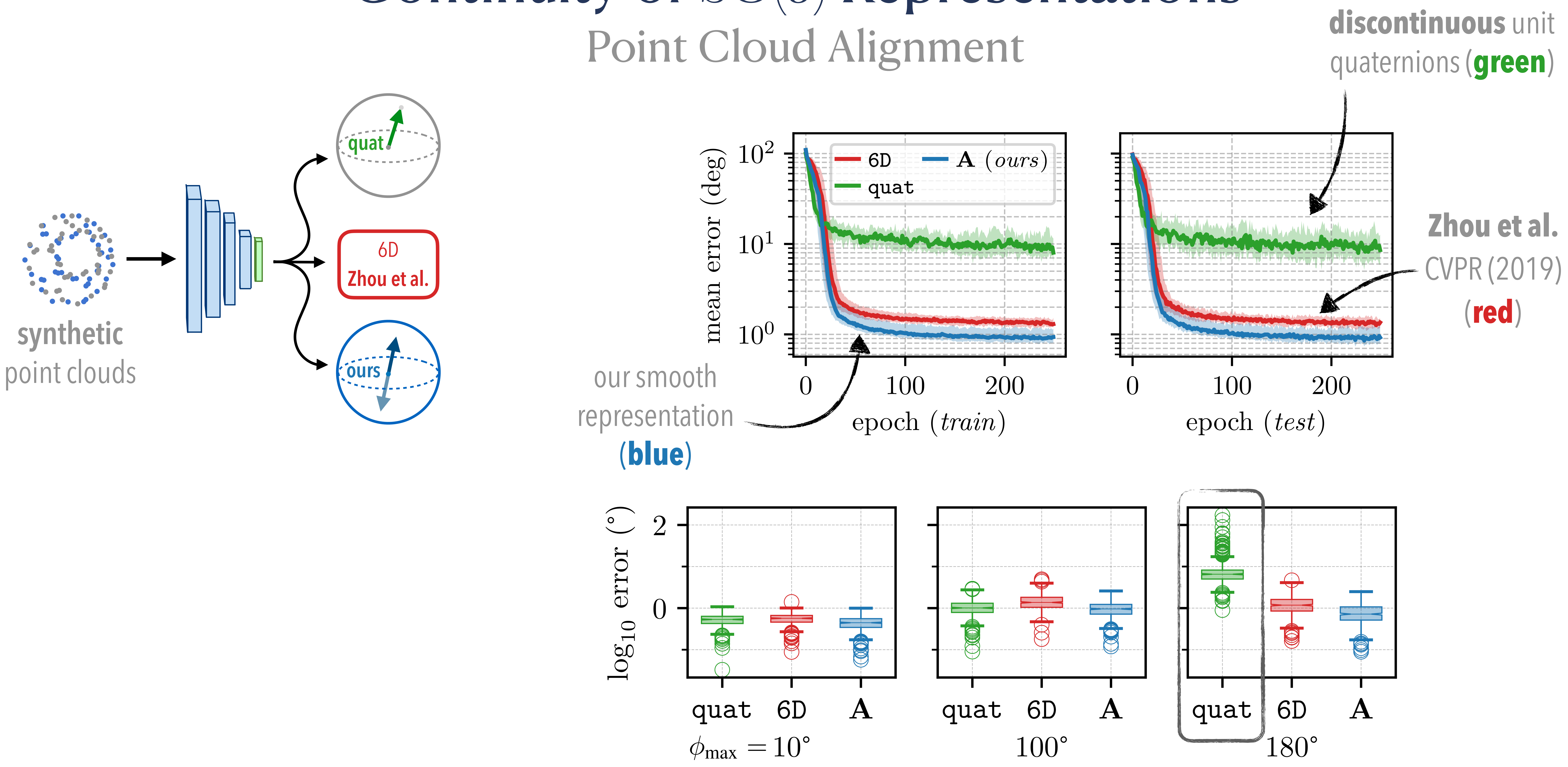


Our representation admits a **smooth** g (*continuous and differentiable*)

Theorem 1 (Smooth Global Section, $\mathrm{SO}(3) \to \mathbb{S}^4_\lambda$)**.** *Consider the surjective map $f : \mathbb{S}^4_\lambda \to \mathrm{SO}(3)$ such that $f(\mathbf{A})$ returns the rotation matrix defined by the two antipodal unit quaternions $\pm\mathbf{q}^*$ that minimize Problem 3. There exists a smooth and global mapping, or section, $g : \mathrm{SO}(3) \to \mathbb{S}^4_\lambda$ such that $f(g(\mathbf{R})) = \mathbf{R}$.*

Continuity of $\mathrm{SO}(3)$ Representations

Point Cloud Alignment



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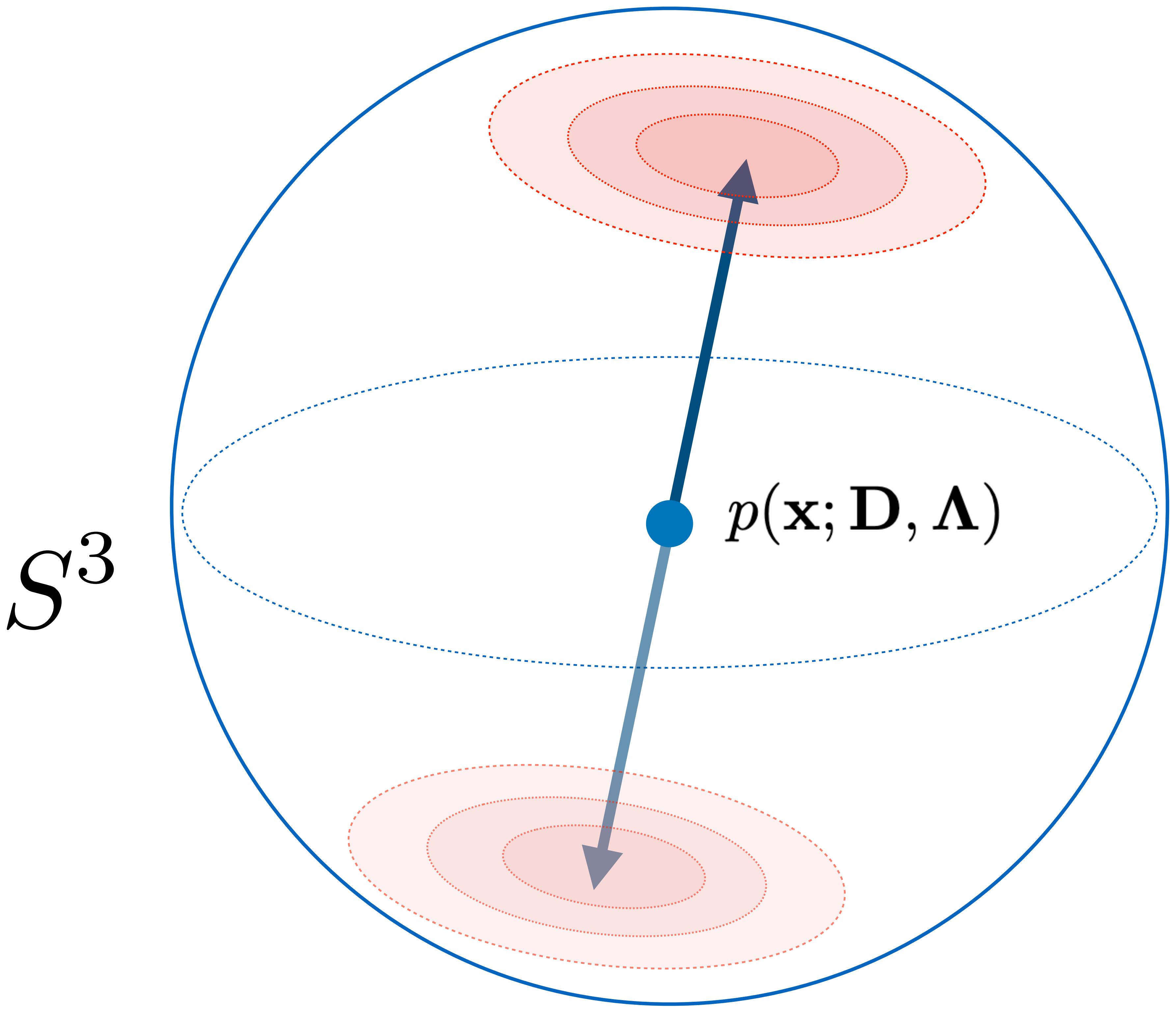
3

1

The *Bingham* Density and $\mathbf{A}(\boldsymbol{\theta})$



anti-podally symmetric

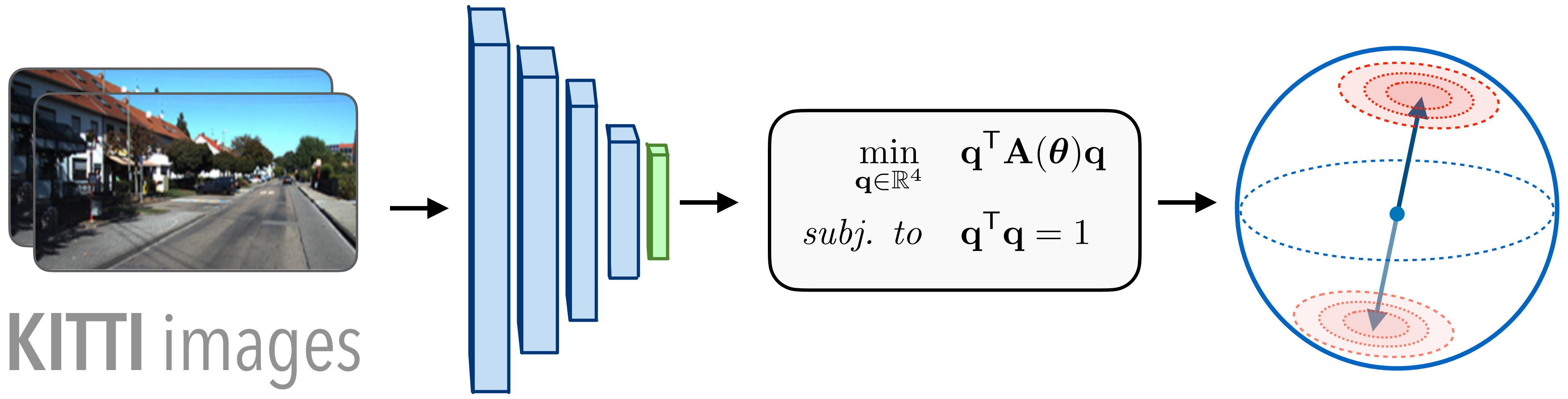
$$p(\mathbf{x}; \mathbf{D}, \boldsymbol{\Lambda}) = \frac{1}{N(\boldsymbol{\Lambda})} \exp\left(\mathbf{x}^\top \mathbf{D} \boldsymbol{\Lambda} \mathbf{D}^\top \mathbf{x}\right)$$

4x4 symmetric $\mathbf{A}$ defines a Bingham *belief*

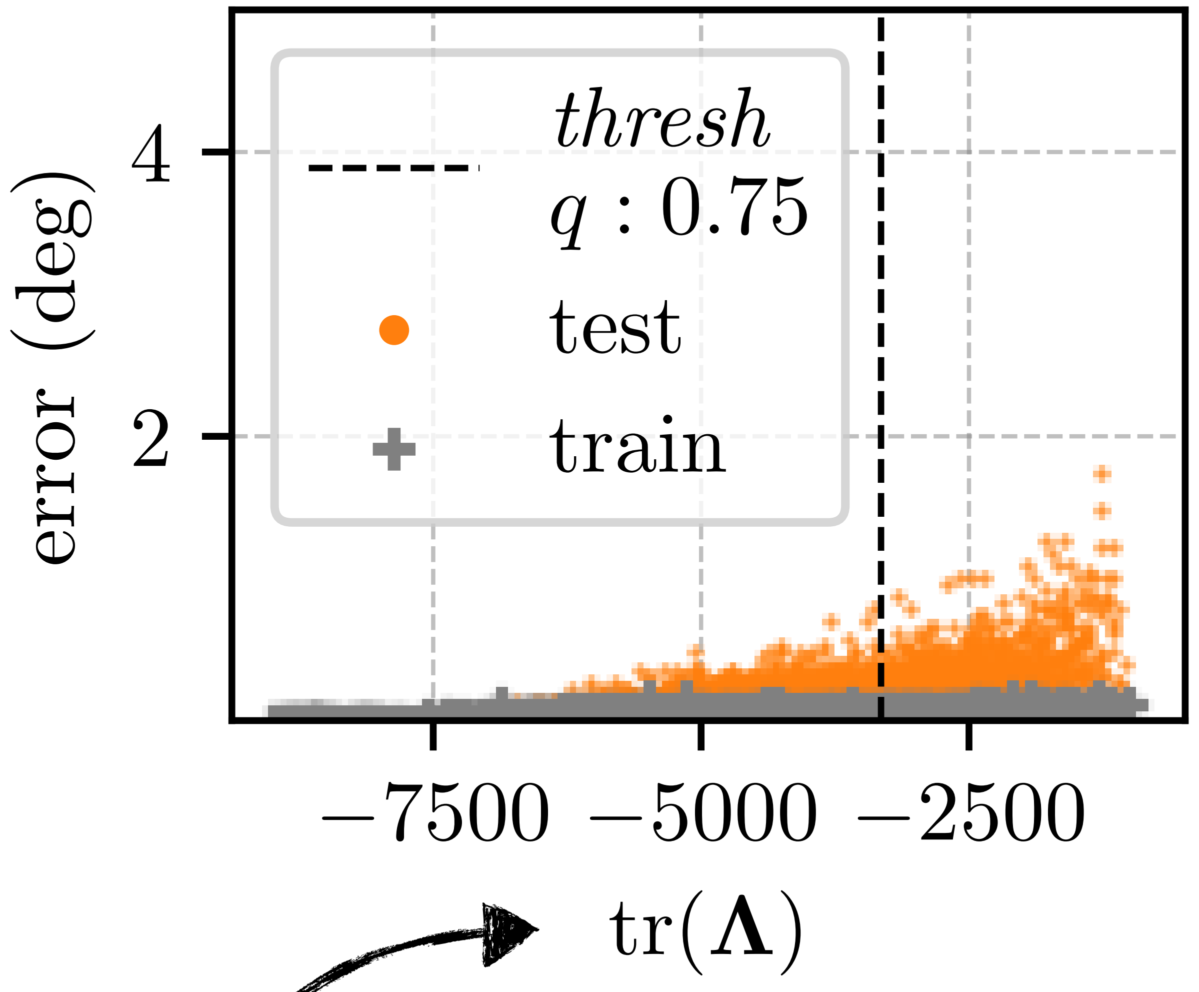
$$\mathbf{A}(\boldsymbol{\theta}) \iff \mathbf{D} \boldsymbol{\Lambda} \mathbf{D}^\top$$

Out-of-Distribution Detection

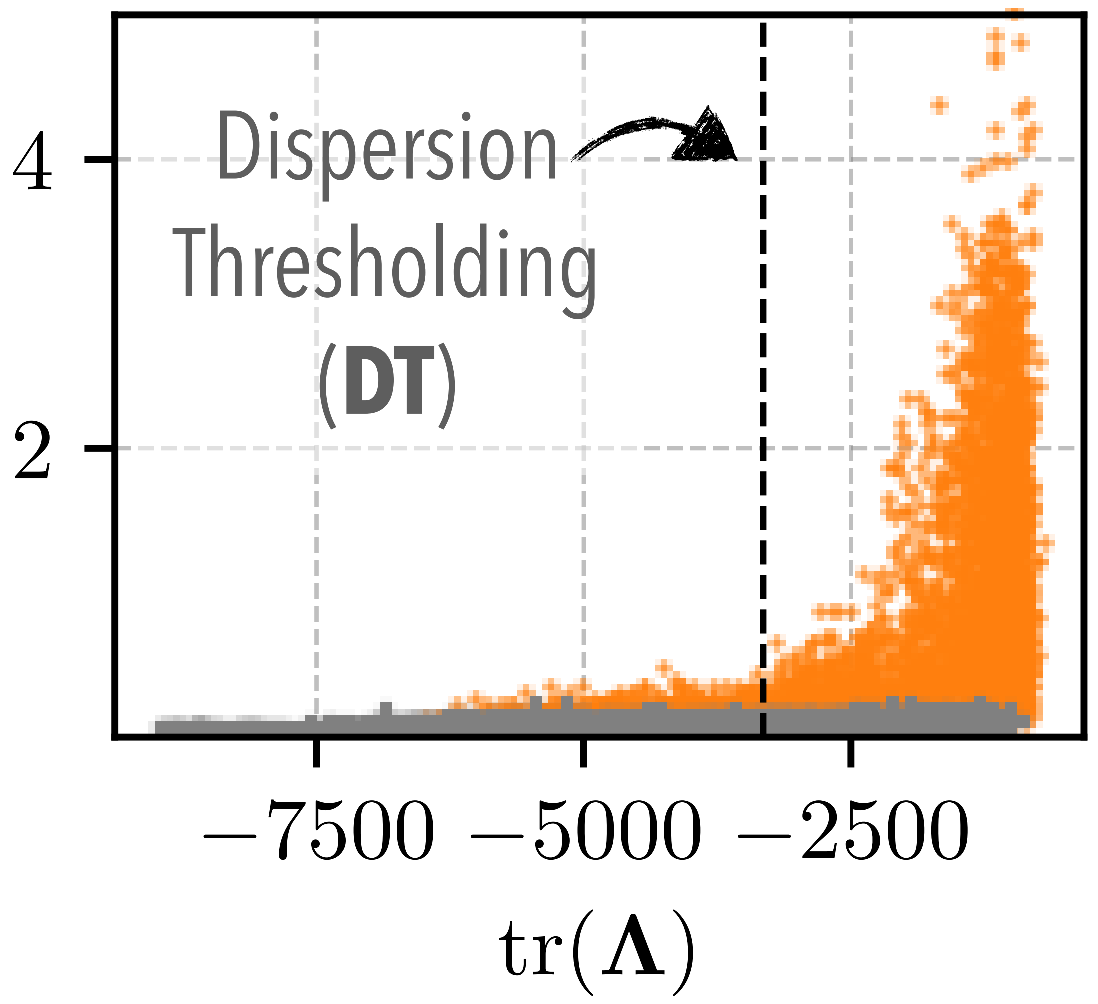
Robust Relative Rotation from Images



KITTI images



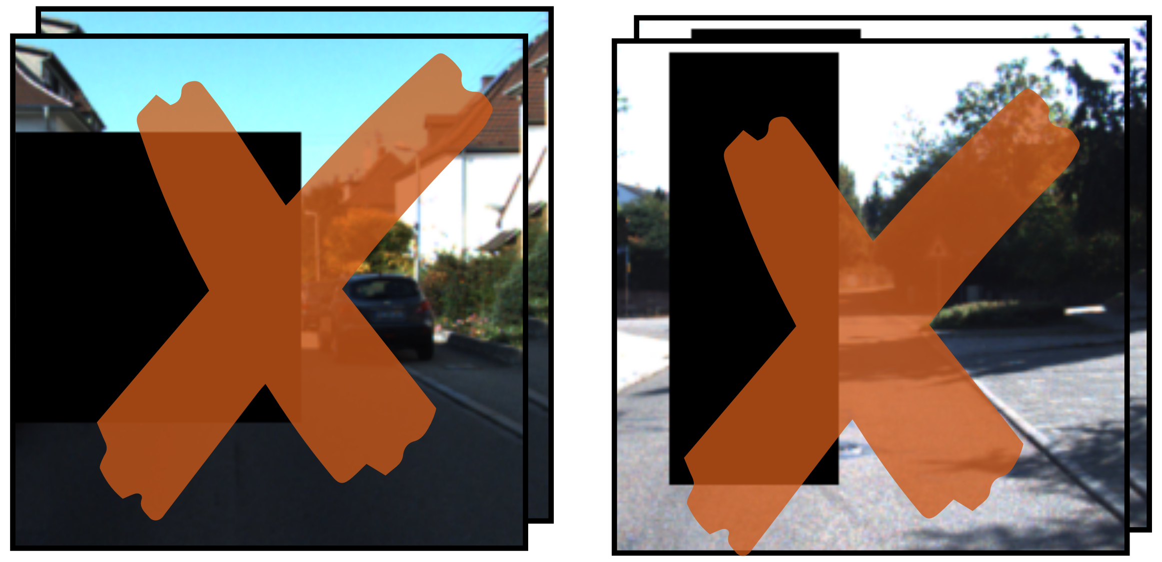
corrupted test set



Dispersion Coefficient

based on eigenvalues of learned $\mathbf{A}$

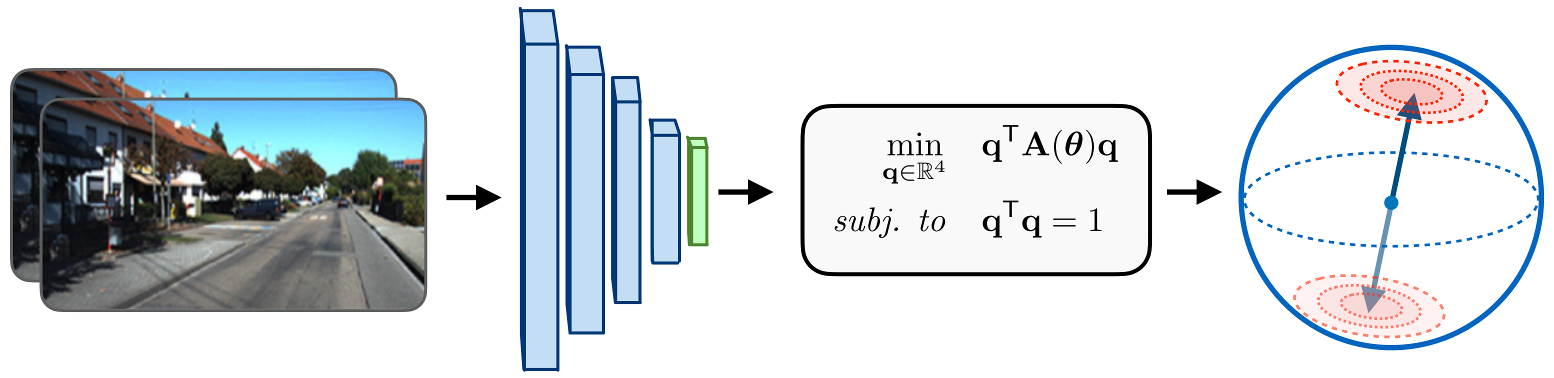
$$\mathrm{tr}\left(\mathbf{\Lambda}\right) = 3\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, \ \lambda_i \in \lambda(\mathbf{A})$$

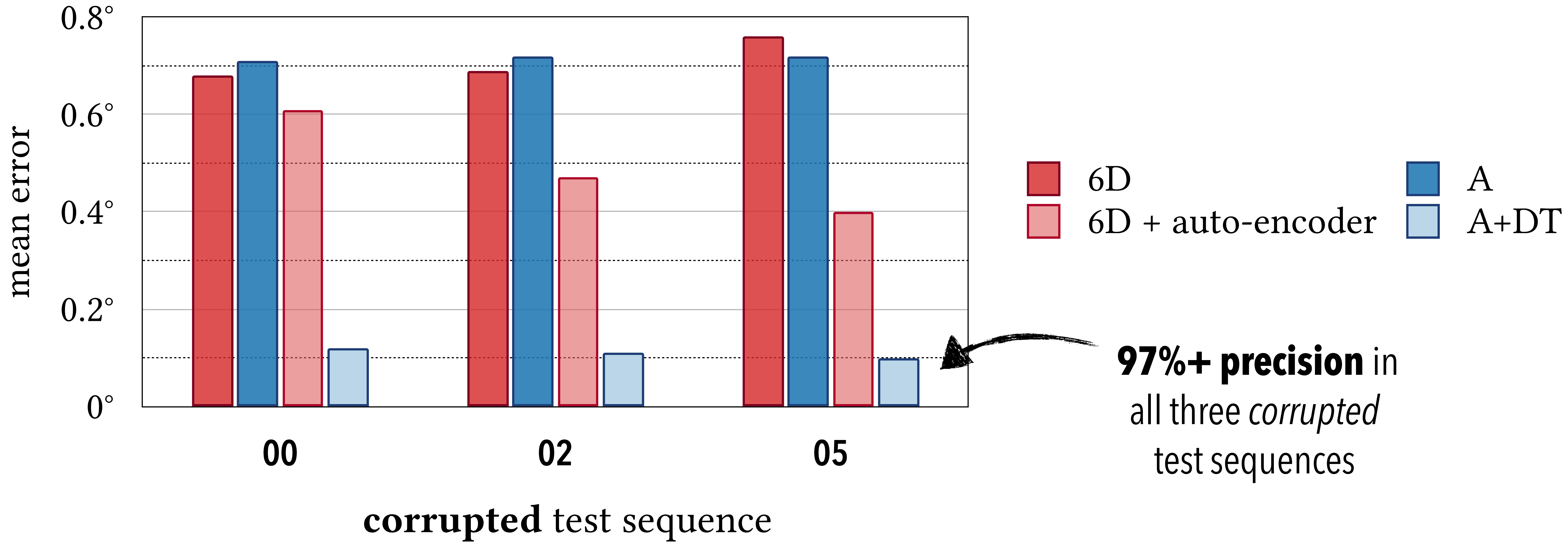


DT rejects OOD inputs

Out-of-Distribution Detection

Robust Relative Rotation from Images



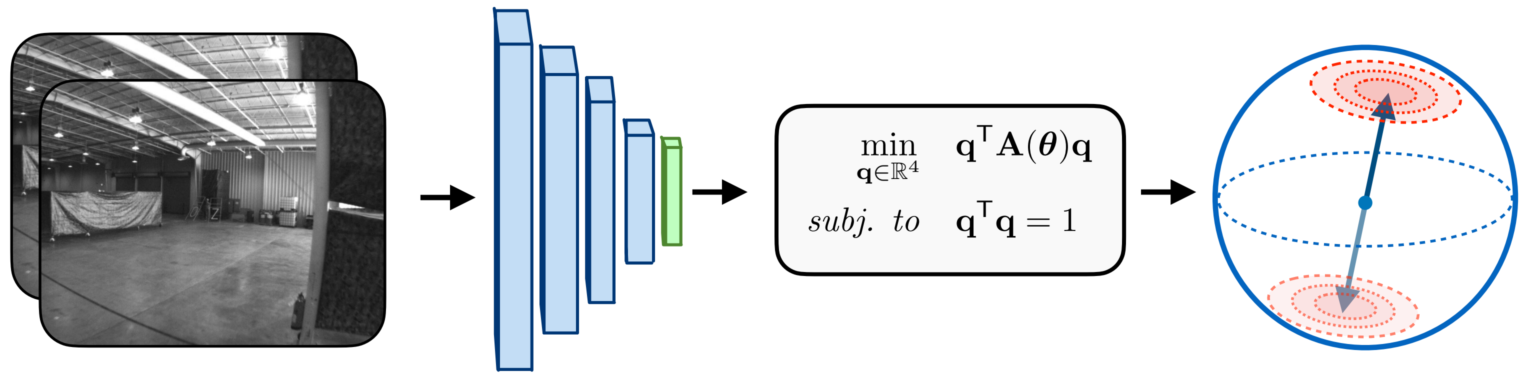


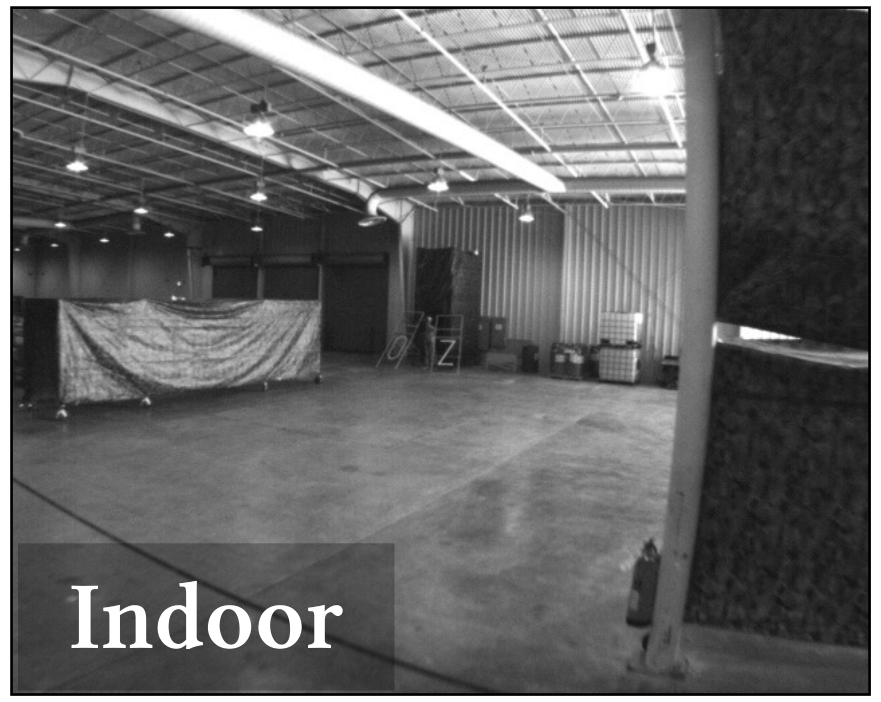
97%+ precision in all three *corrupted* test sequences

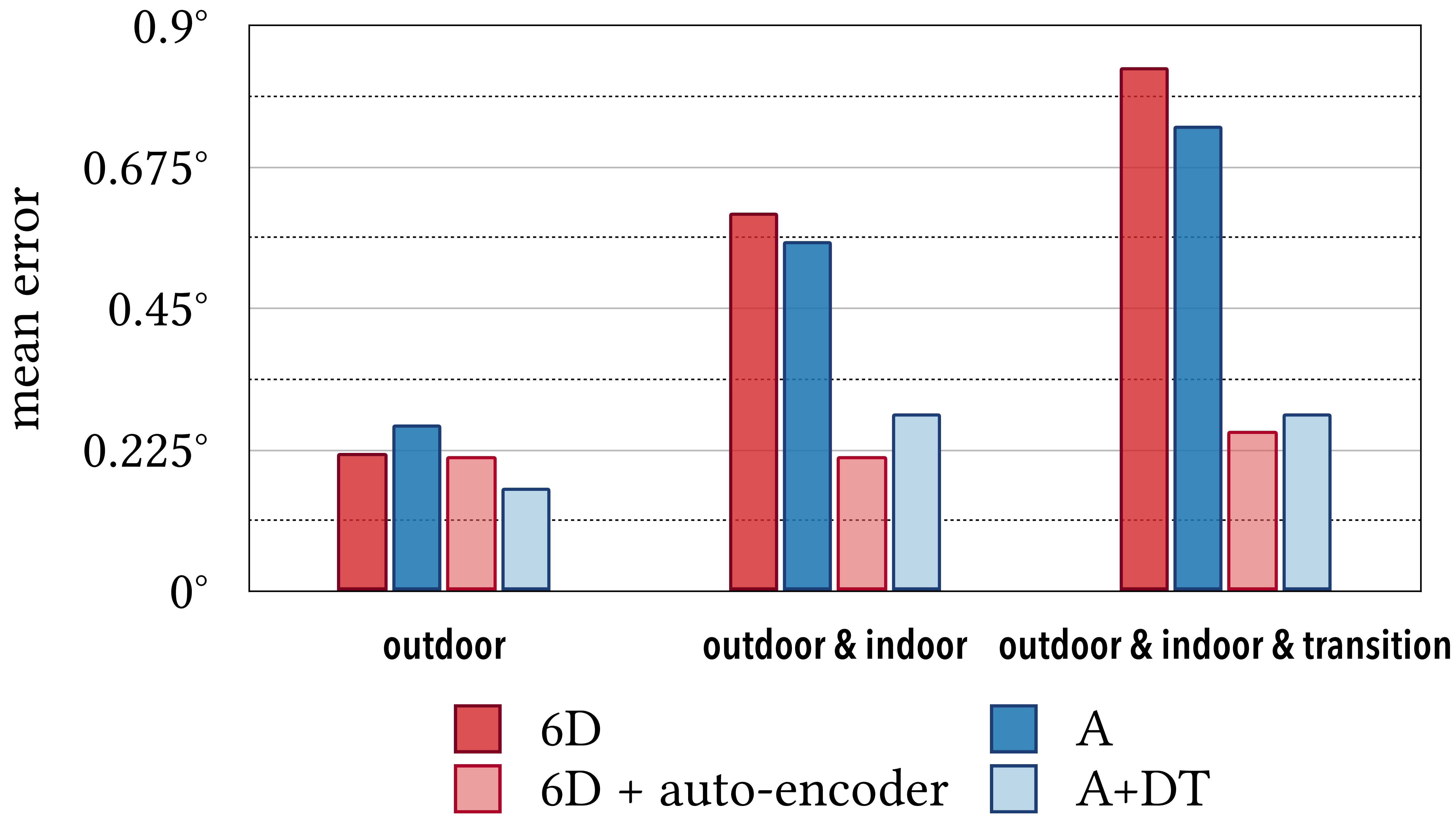
Out-of-Distribution Detection

Robust Relative Rotation from Images

MAV with global shutter camera







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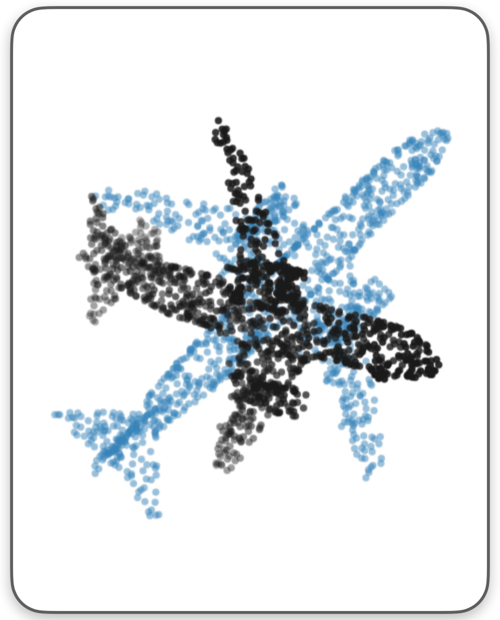
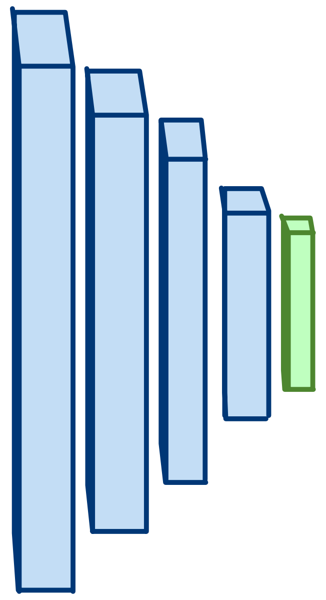
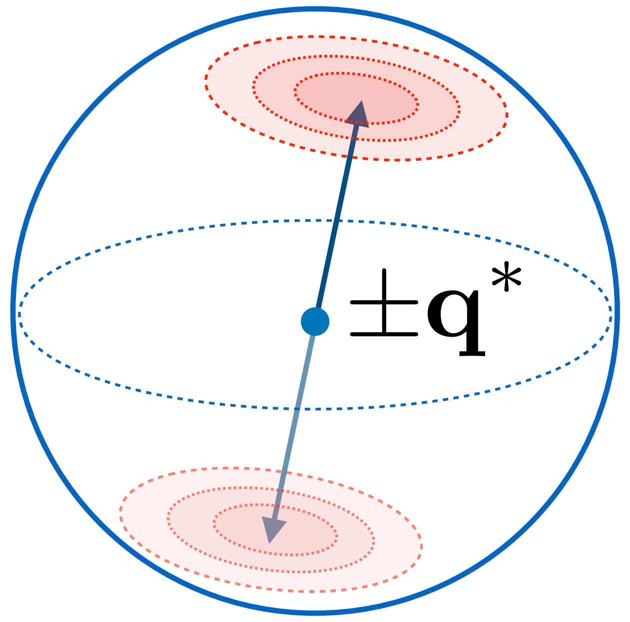
github.com/utiasSTARS/bingham-rotation-learning





$\xrightarrow{\mathbf{A}(\boldsymbol{\theta})}$

$$\begin{array}{ll} \min\limits_{\mathbf{q} \in \mathbb{R}^4} & \mathbf{q}^\intercal \mathbf{A}(\boldsymbol{\theta}) \mathbf{q} \\ \textit{subj. to} & \mathbf{q}^\intercal \mathbf{q} = 1 \end{array}$$



Smooth
useful for large rotation targets

Encodes Bingham belief

Easy to implement
a few lines in PyTorch

Can slow down training
differentiable layer requires an eigendecomposition and linear solve

OOD mechanism
further investigation required

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