Certifiably Correct SLAM

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Motivation

Robotics: The science of *physically embodied* autonomous agents.

Core capabilities:

- Planning
- Navigation
- Manipulation ...

$\Rightarrow$ These require *understanding the world's geometry*.

Example: Autonomous vehicles

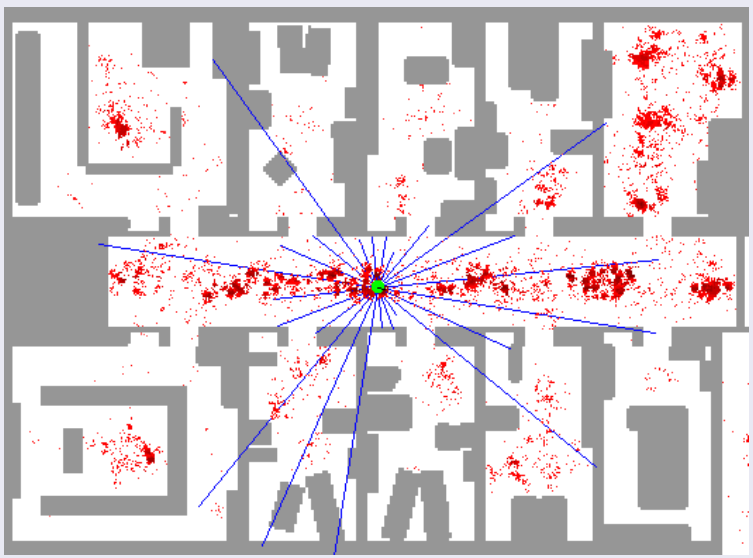
Video credit: Google

Two fundamental problems in robotic spatial perception

Localization ("Where am I?")

Given: Environmental model

Estimate: Robot pose

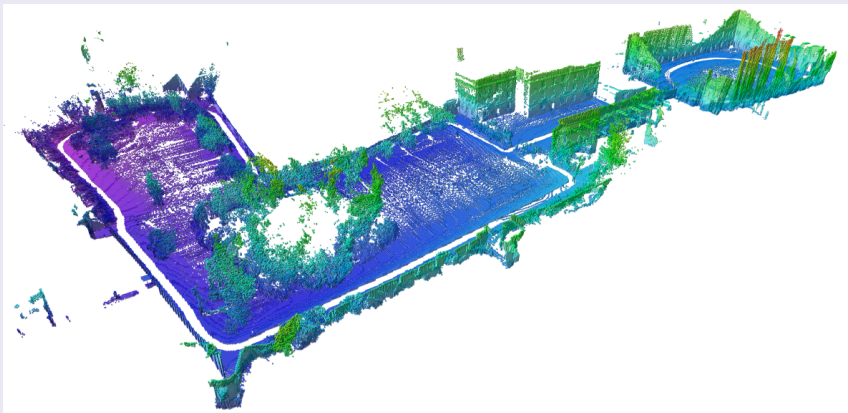


Fox et al. 1999

Mapping ("What's around me?")

Given: Robot pose

Estimate: Environmental model

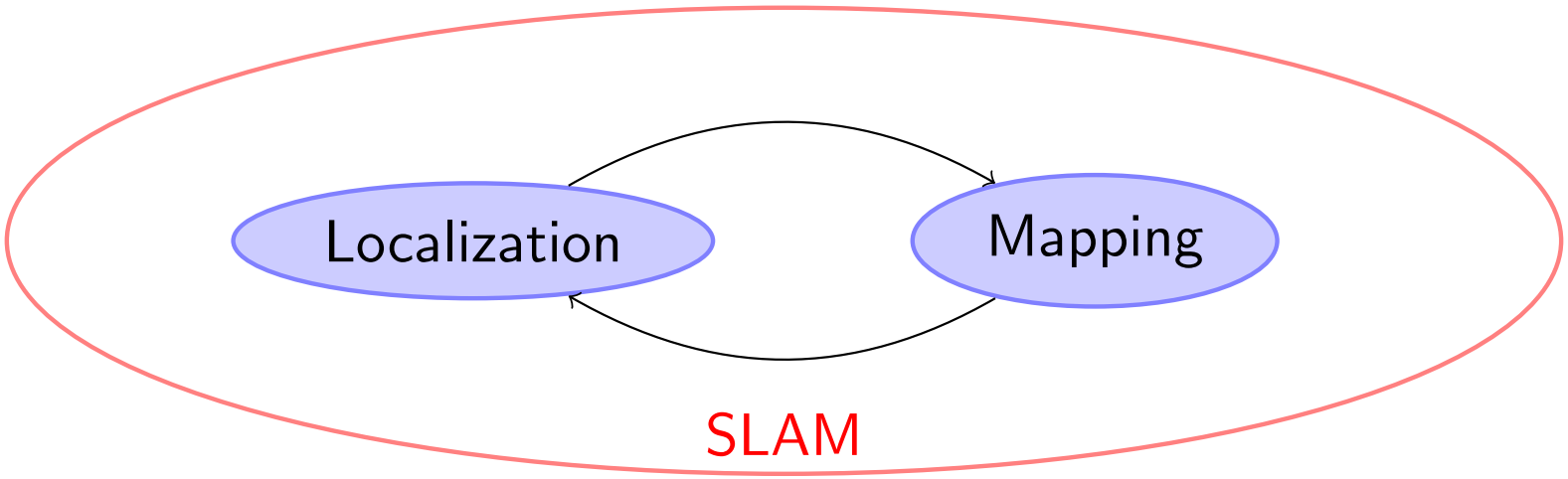


Hornung et al. 2013

$\Rightarrow$ Given *either* pose or map, we can find the other via *recursive Bayesian estimation*.

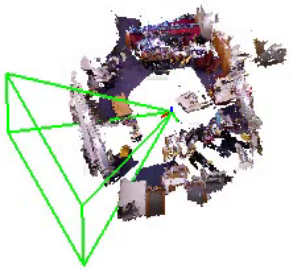
Simultaneous localization and mapping (SLAM)

Given **neither** robot pose nor map, *jointly estimate both*.



- **Much** harder than localization or mapping alone.
- Enables operation in *unknown environments*.
- An *essential enabling technology* for mobile robots.

Simultaneous localization and mapping (SLAM)



H. Johannsson et al. "Temporally Scalable Visual SLAM Using a Reduced Pose Graph". In: *IEEE Intl. Conf. on Robotics and Automation (ICRA)*. Karlsruhe, Germany, May 2013, pp. 54–61

In this talk

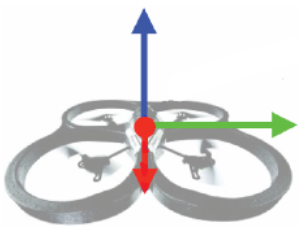
1. Lightning introduction to pose-graph SLAM
   - Problem formulation
   - The problem of *nonconvexity*

2. Our contributions: Certifiably correct SLAM
   - A *convex relaxation* for pose-graph SLAM
   - A fast optimization method to solve this relaxation efficiently

Payoff: SE-Sync, a *certifiably correct* algorithm for SLAM

A concrete example

Consider a robot exploring some initially unknown environment...



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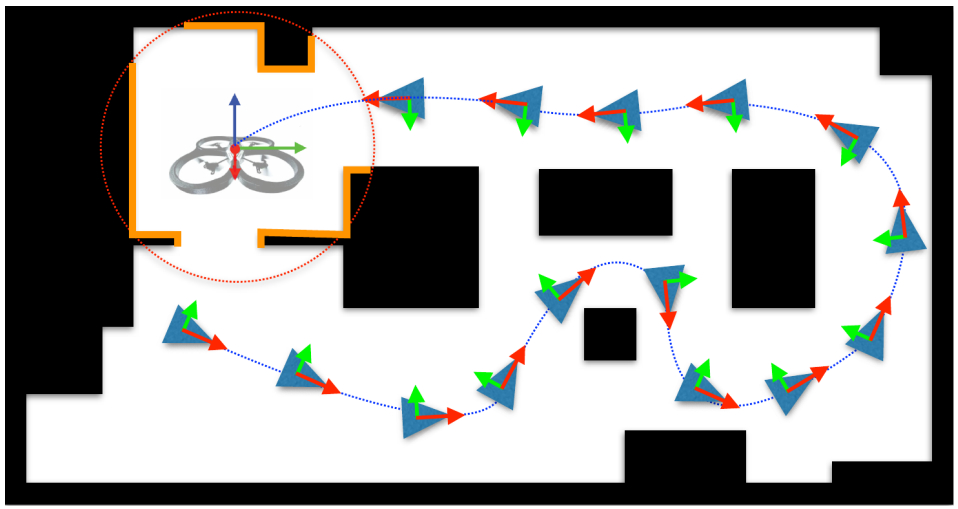

Figures courtesy of Luca Carlone

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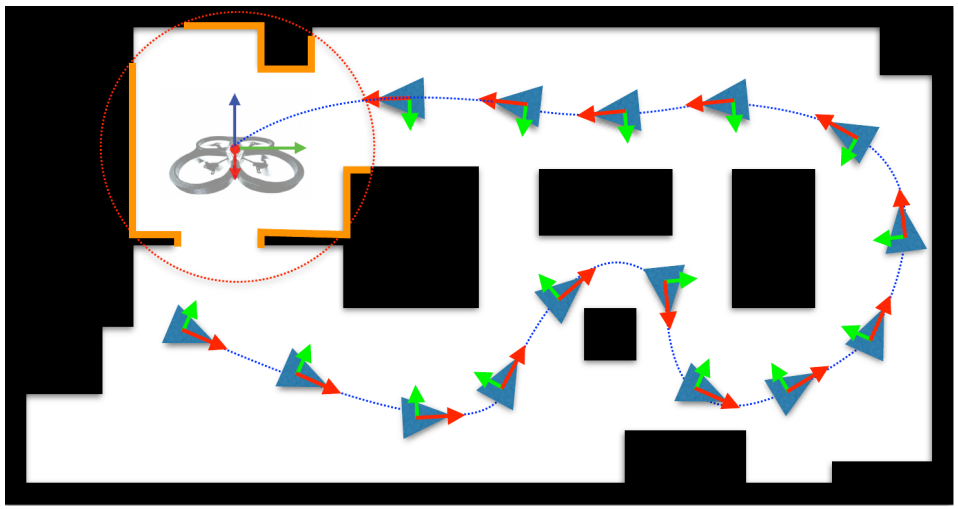

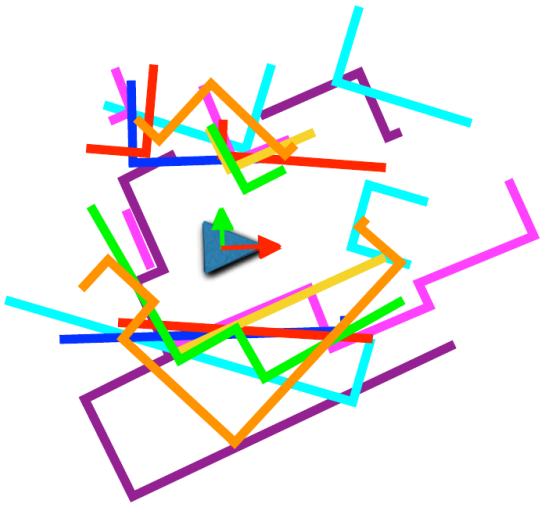

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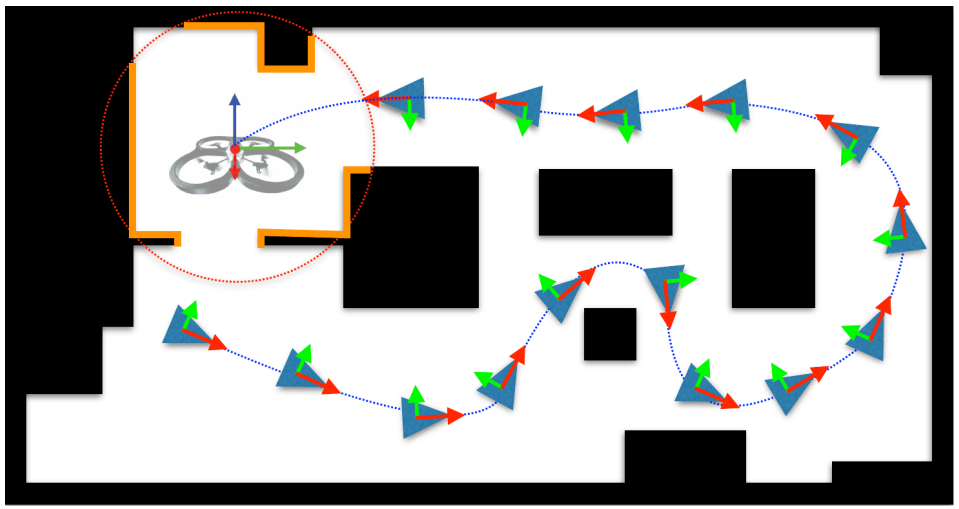

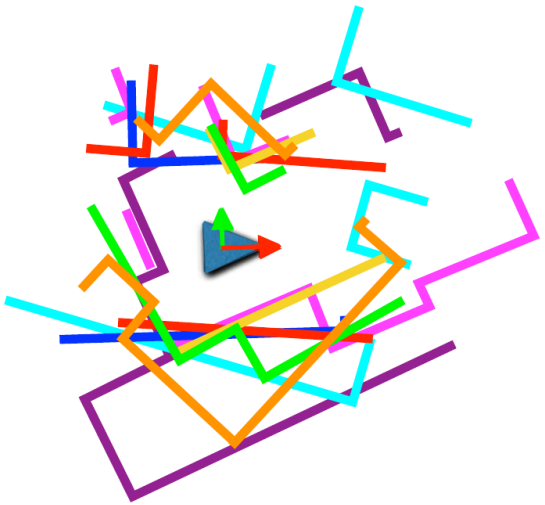

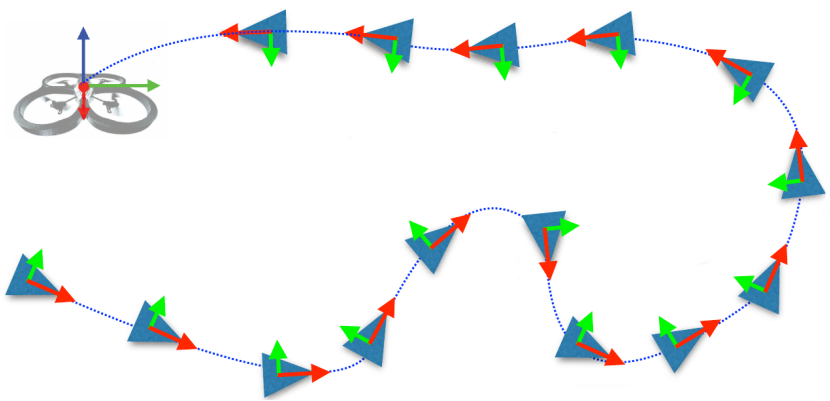

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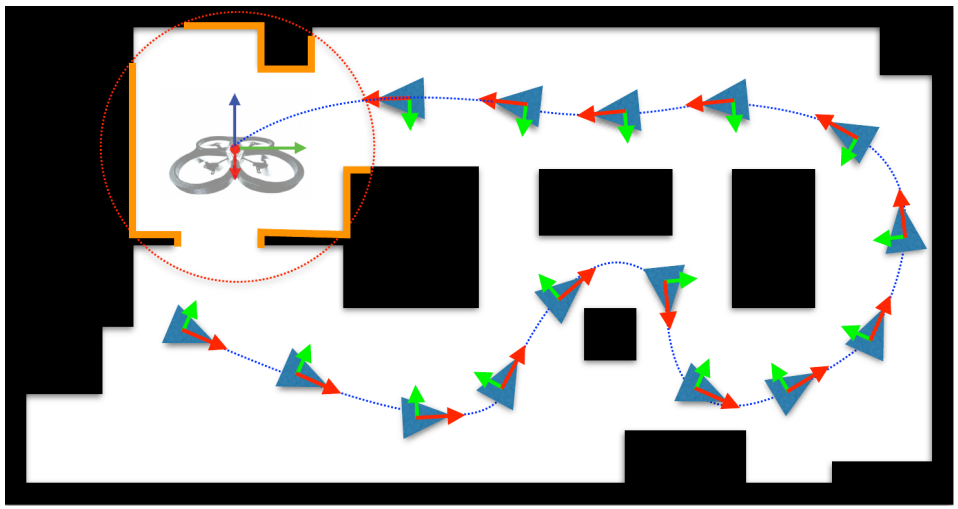

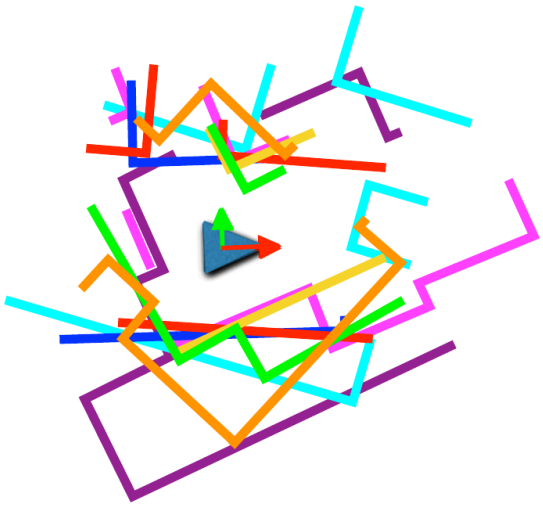

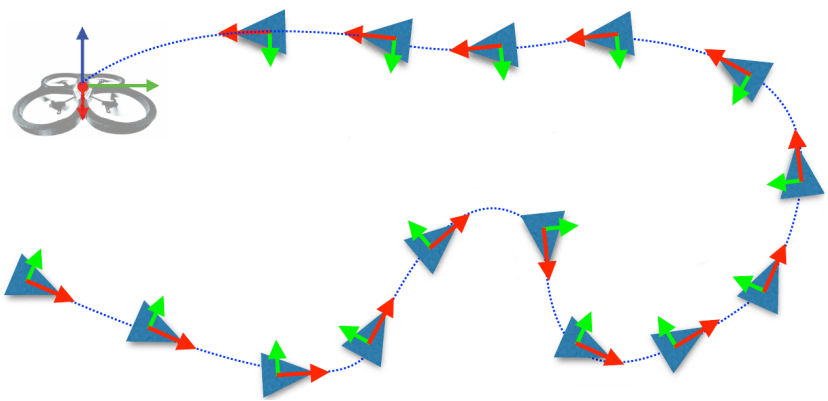

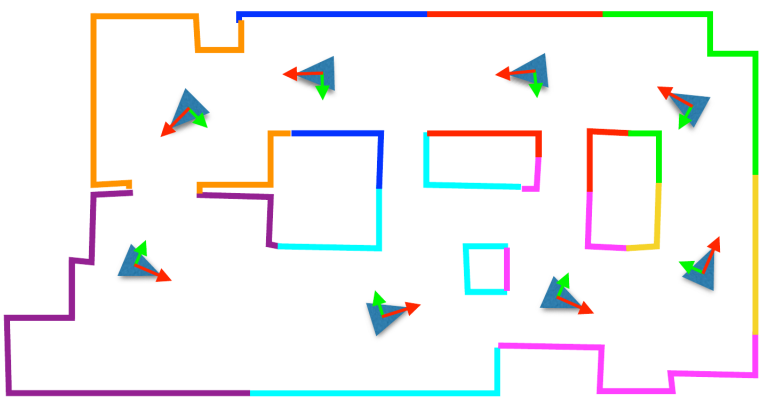

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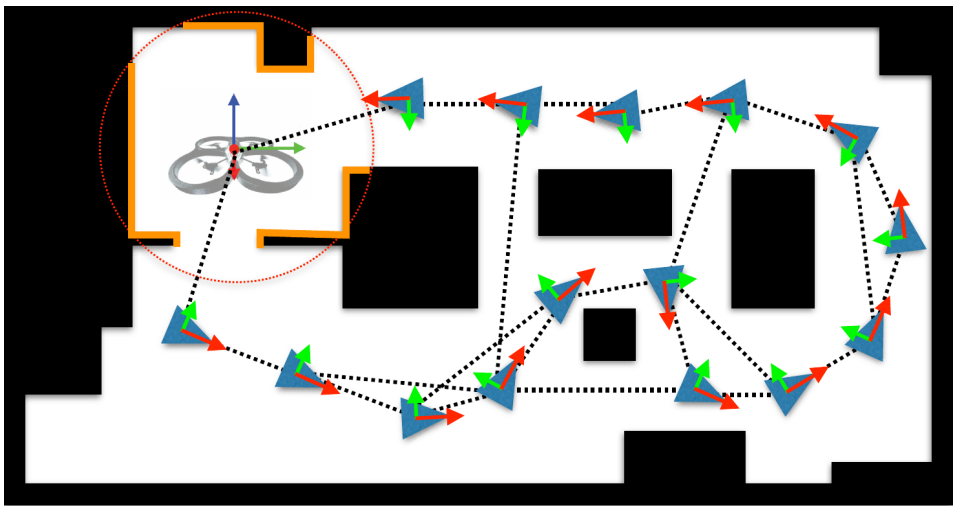
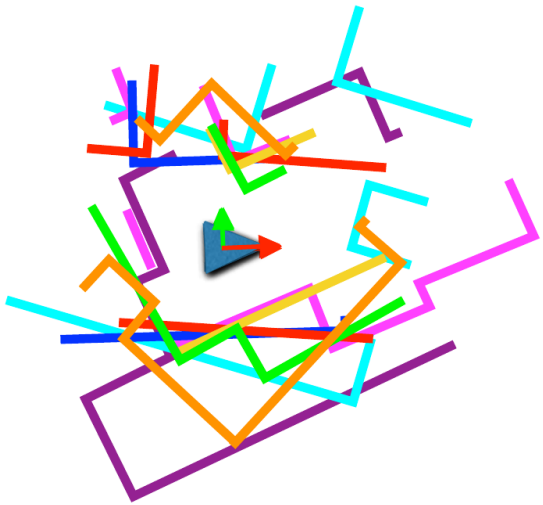
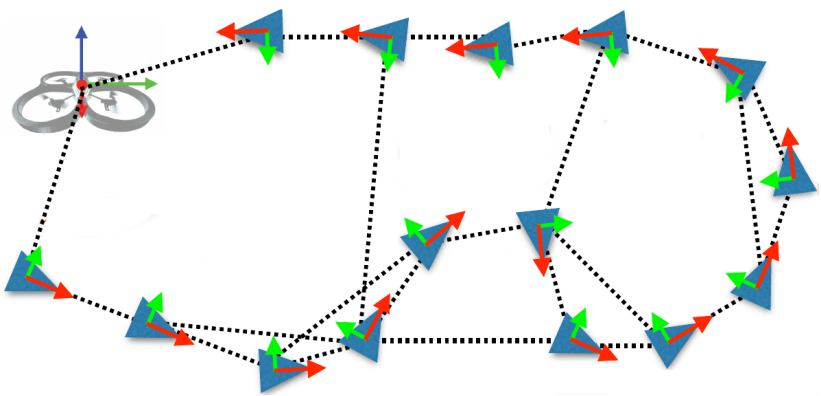
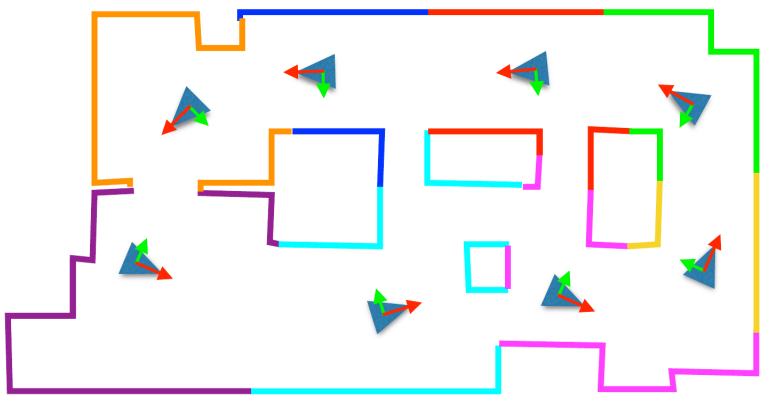




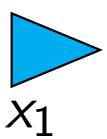




Figures courtesy of Luca Carlone

The network of spatial relations

More abstractly, as the robot explores...



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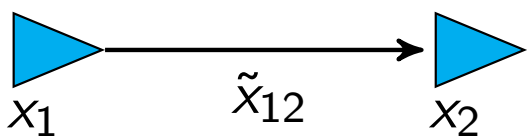

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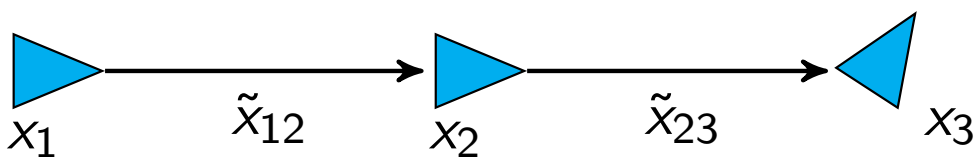

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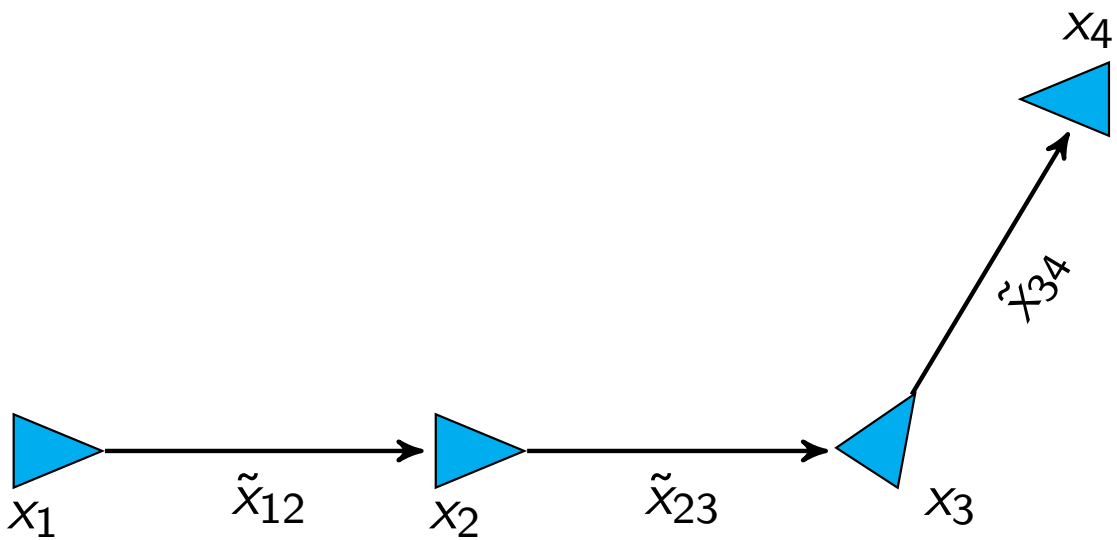

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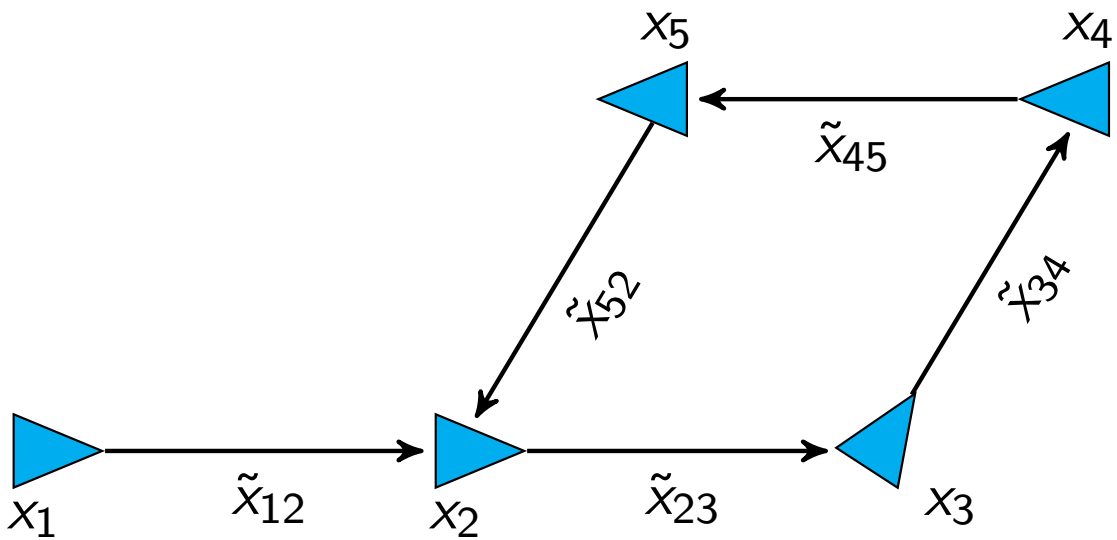

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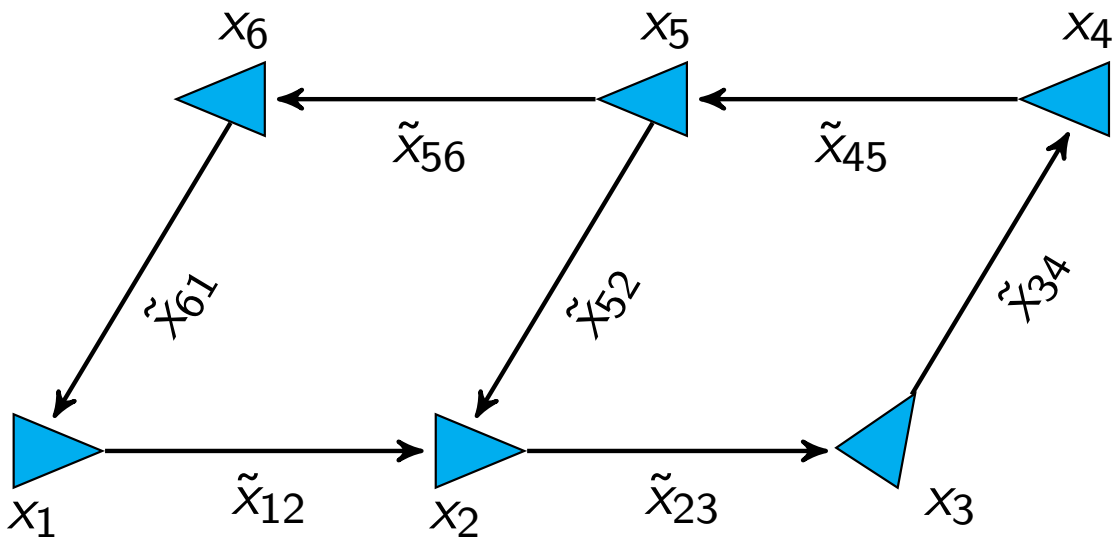

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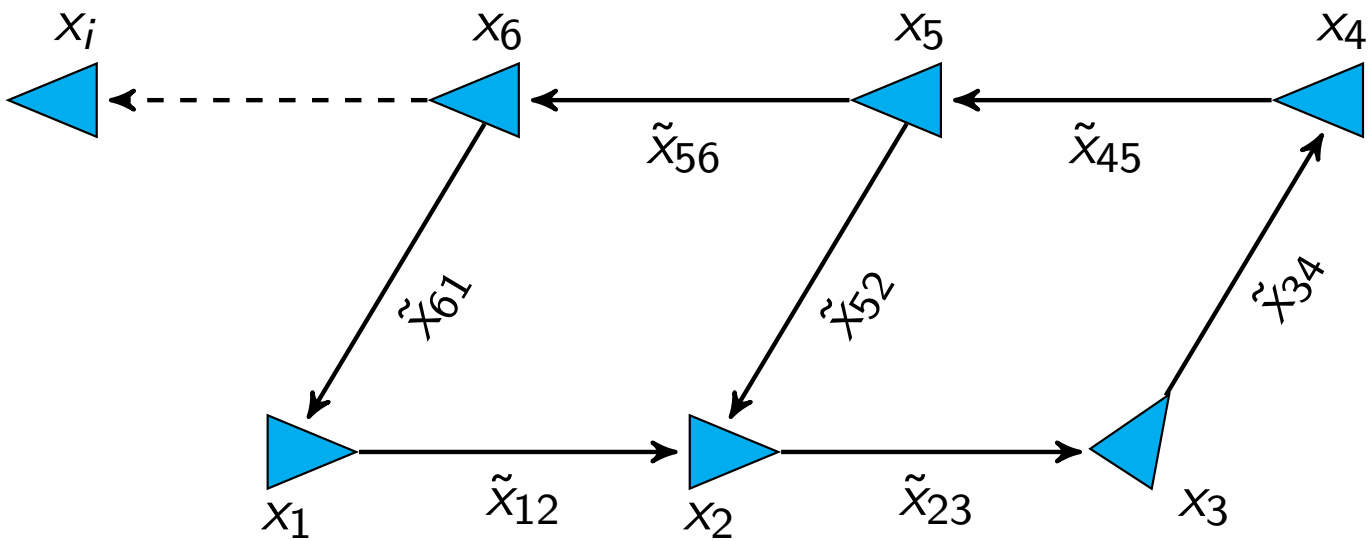

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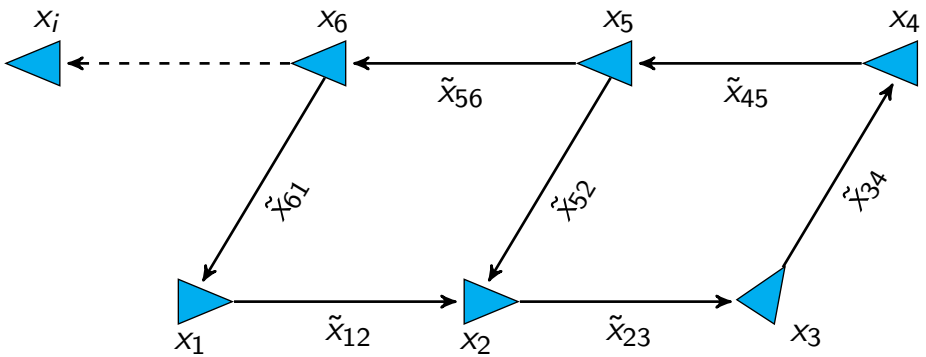
Formalizing the problem: Pose graphs

A *pose-graph* is a directed graph $G = (X, \vec{\mathcal{E}})$ that models the *joint distribution* $p(\tilde{\mathcal{Y}}|X)$ of this network of noisy relative measurements:



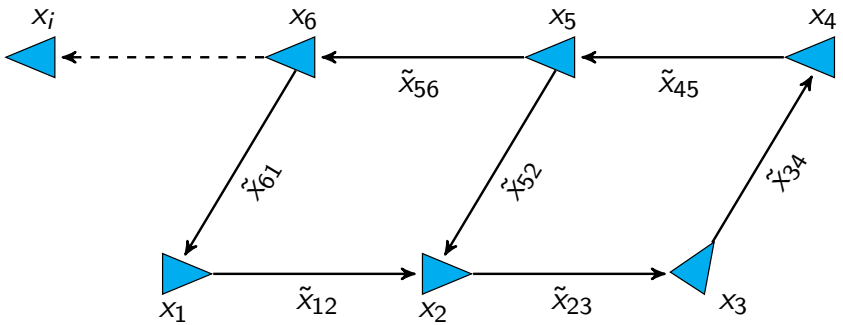
$$p(\tilde{\mathcal{Y}}|X) = \prod_{(i,j)\in\vec{\mathcal{E}}} p_{ij}(\tilde{x}_{ij}|x_i, x_j)$$

Here:

- $X = \{x_1, \ldots, x_n\}$ are the (latent) *robot poses*
- $\tilde{\mathcal{Y}} = \{\tilde{x}_{ij}\}$ are the *measurements* of relative transforms $x_i^{-1} x_j$
- $\tilde{x}_{ij} \sim p_{ij}(\cdot|x_i, x_j)$ is the *measurement model* for $\tilde{x}_{ij}$.

The pose-graph SLAM problem

Given: A pose-graph model $G = (X, \vec{\mathcal{E}})$:



$$p(\tilde{\mathcal{Y}}|X) = \prod_{(i,j)\in\vec{\mathcal{E}}} p_{ij}(\tilde{x}_{ij}|x_i, x_j)$$

Find: A *maximum-likelihood estimate* $\hat{X}_{\mathrm{MLE}} \in \mathrm{SE}(d)^n$ for X.

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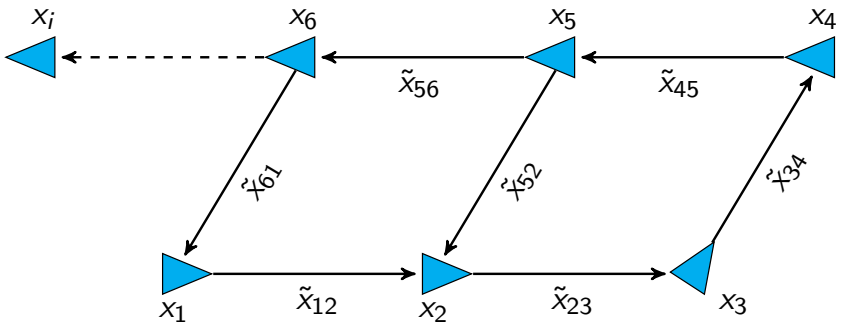

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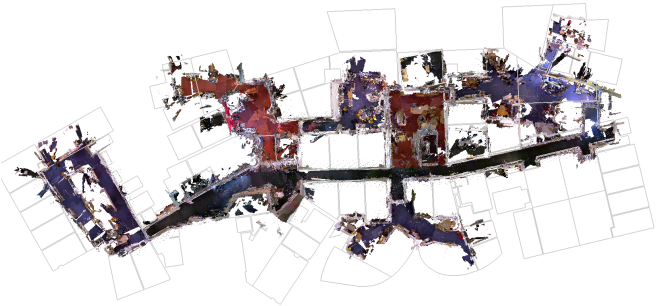
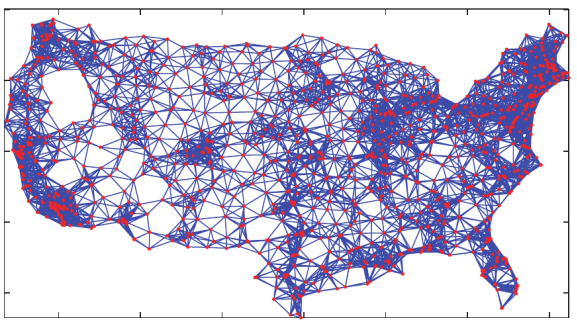
The pose-graph SLAM problem

Many important examples in geometric estimation:

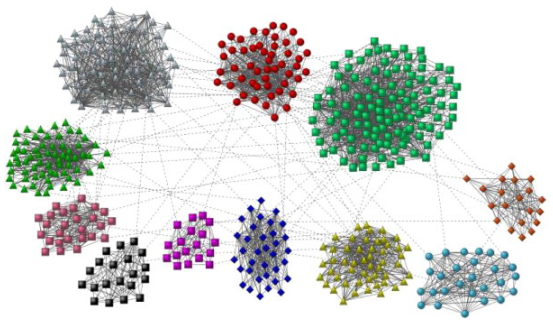
- Robotic mapping
- Camera motion estimation
- Sensor network localization





Special cases:

- Rotation averaging
- Community detection



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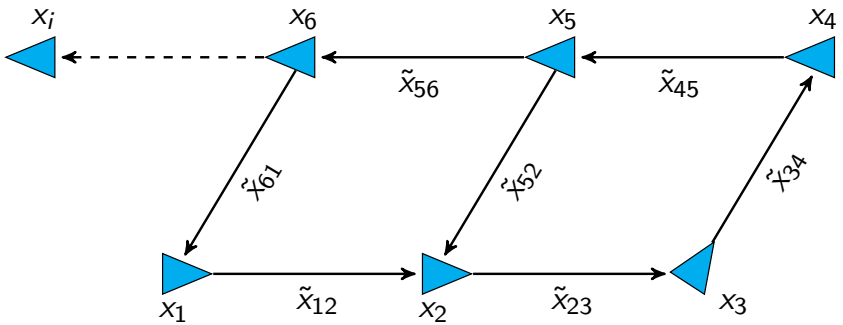

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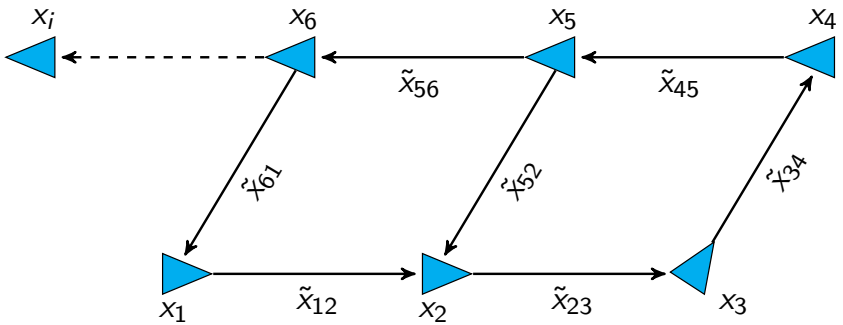

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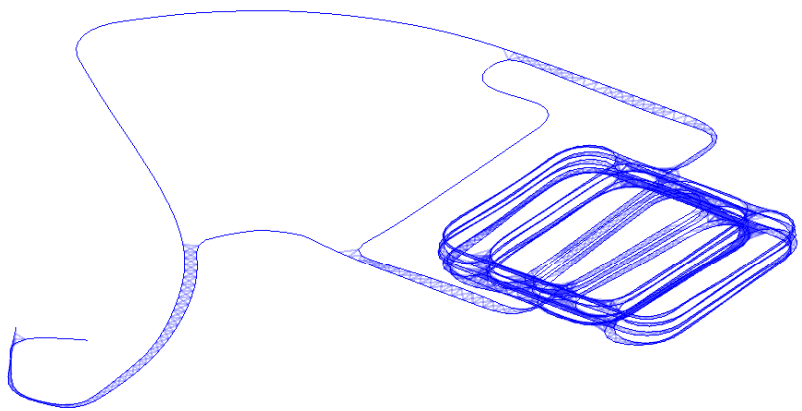
The challenge: This is a *high-dimensional*, *nonconvex* problem
$\Rightarrow$ **LOTS** of (bad) *local minima*

The problem of nonconvexity

The problem of nonconvexity



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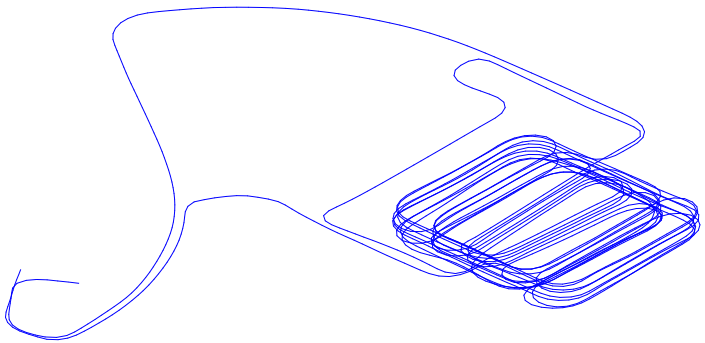
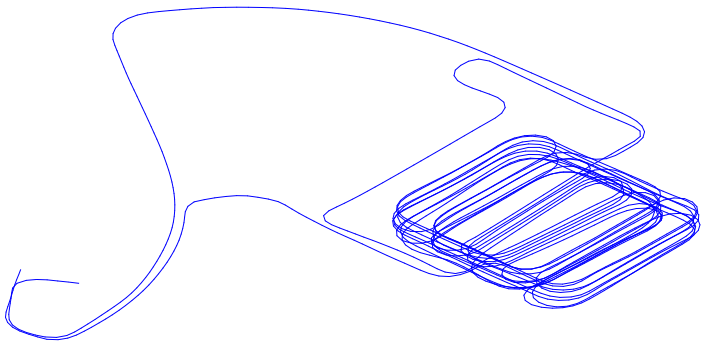


Optimal estimate

The problem of nonconvexity



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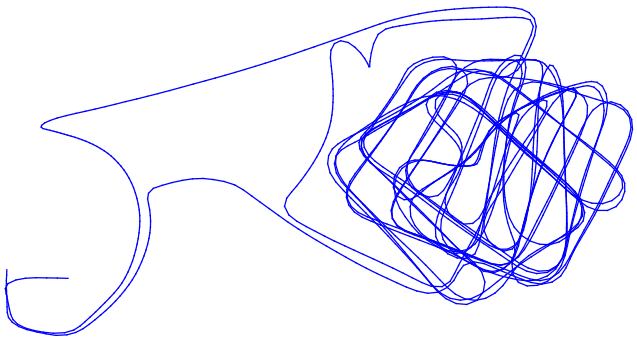
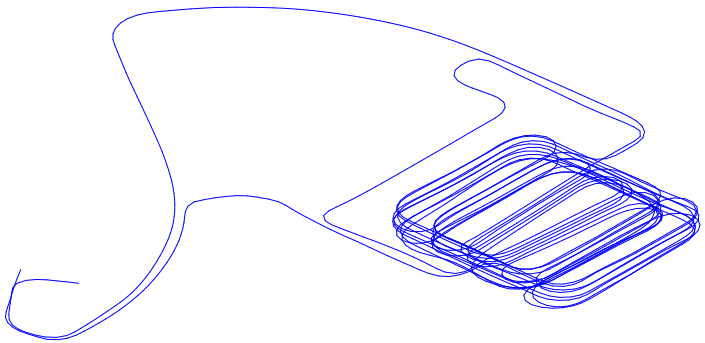


Suboptimal critical point

The problem of nonconvexity



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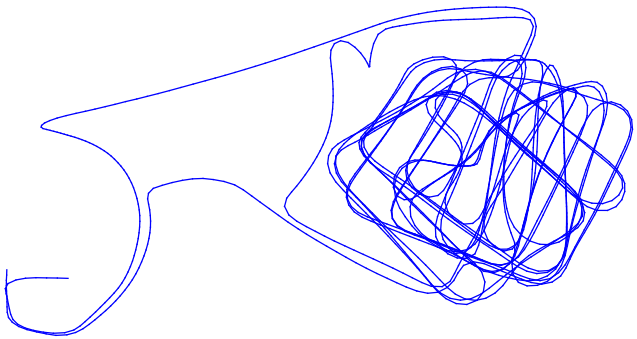

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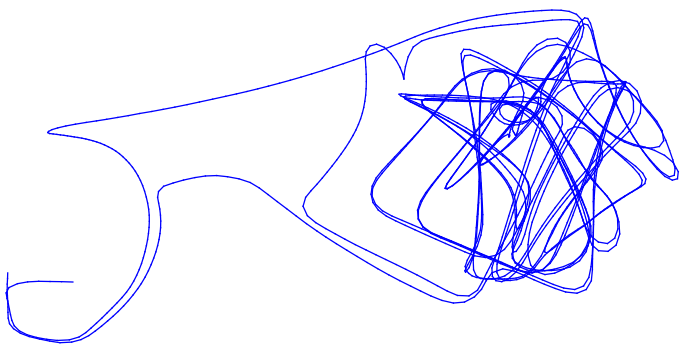

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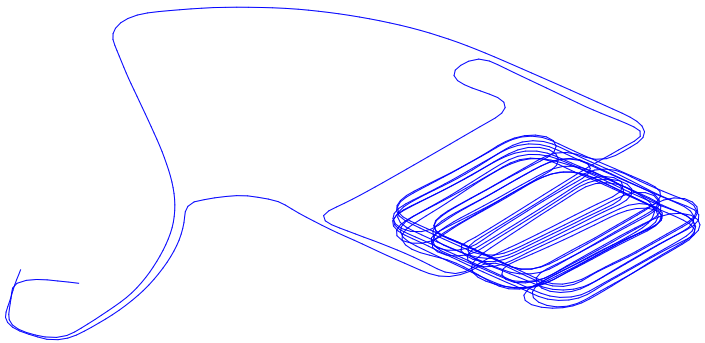

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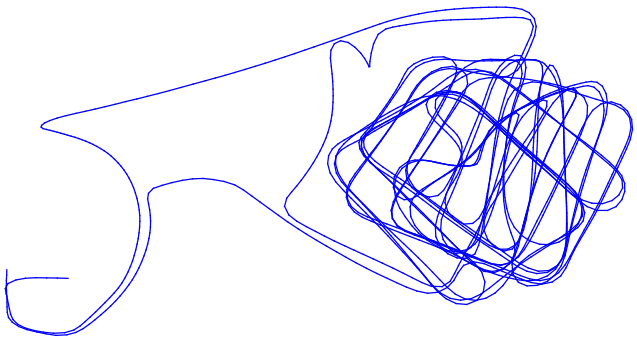

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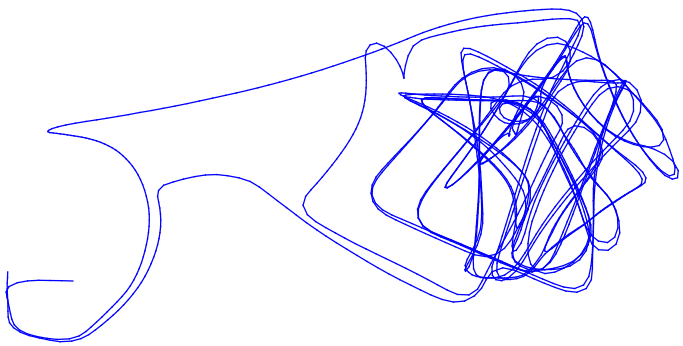

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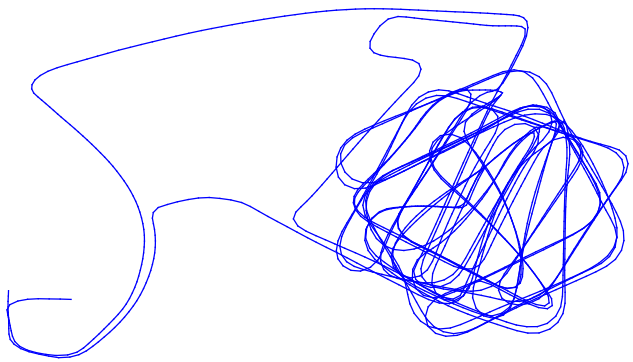
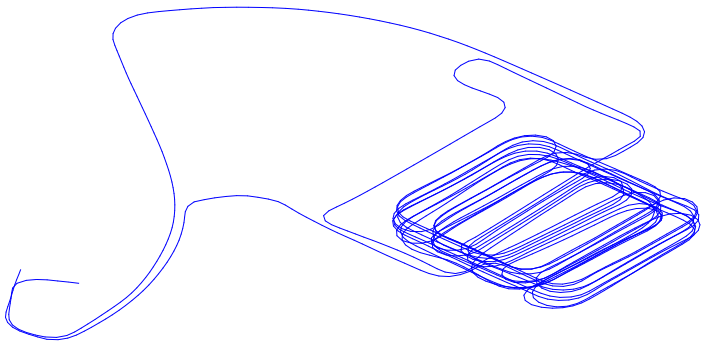


Suboptimal critical point

The problem of nonconvexity

Main question: Can we *ensure* good SLAM performance?



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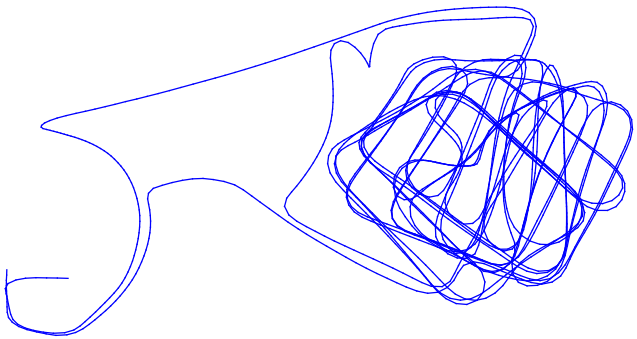

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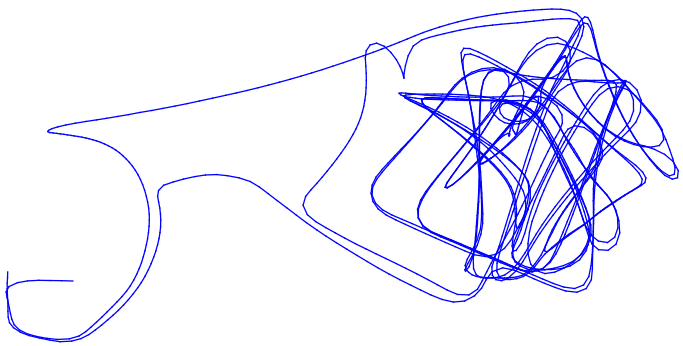

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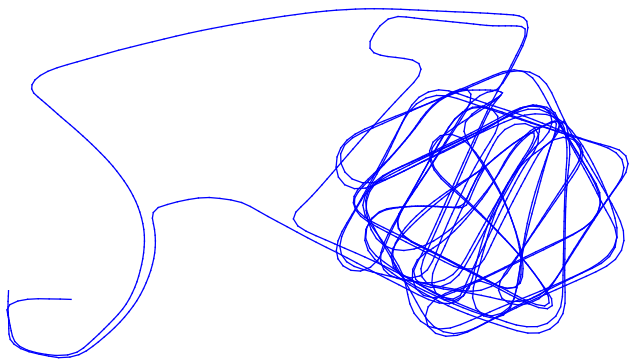

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In the rest of this talk ...

We develop:

- A *convex relaxation* of pose-graph SLAM whose minimizer provides an *exact MLE* for moderate noise
- A fast optimization method to solve this relaxation efficiently

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⇒**SE-Sync**: a *certifiably correct* algorithm for SLAM[1]

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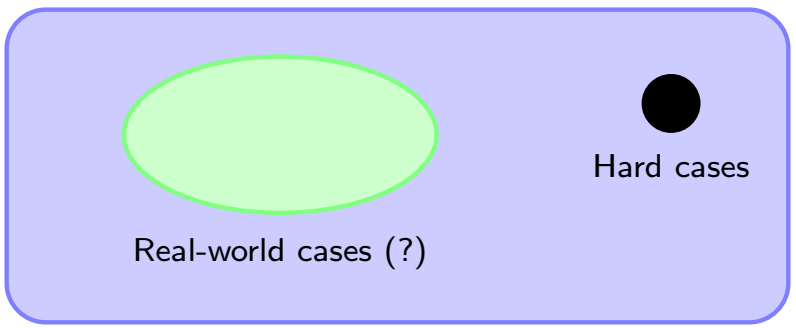

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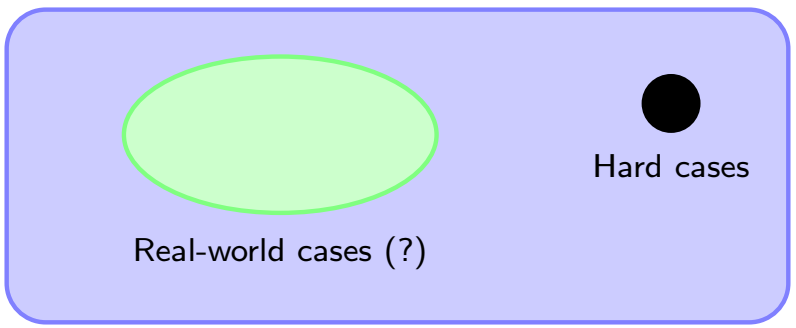

Question: How to *generate* these approximations?

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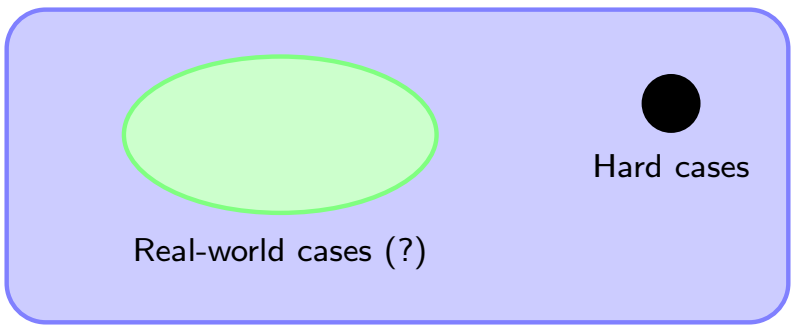

Question: How to *generate* these approximations?

One general approach: *Convex relaxation*

- Standard machinery for constructing these
- Admit (some) formal guarantees on solution quality.[2]

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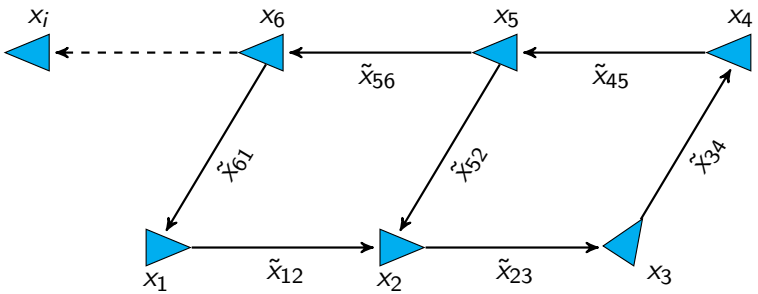
The "standard model" of pose-graph SLAM



$$p(\tilde{\mathcal{Y}}|X) = \prod_{(i,j)\in\vec{\mathcal{E}}} p_{ij}(\tilde{x}_{ij}|x_i, x_j)$$

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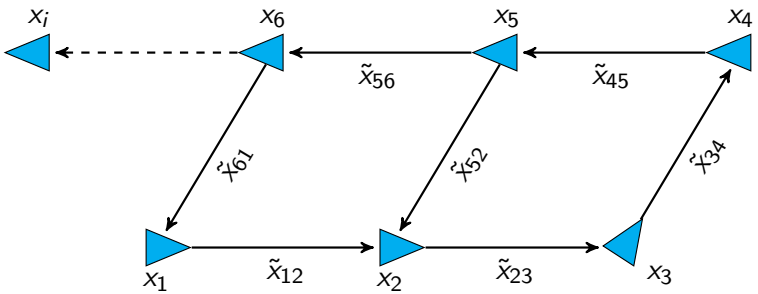

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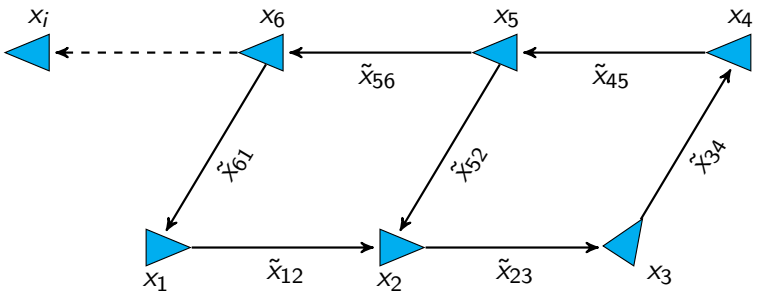

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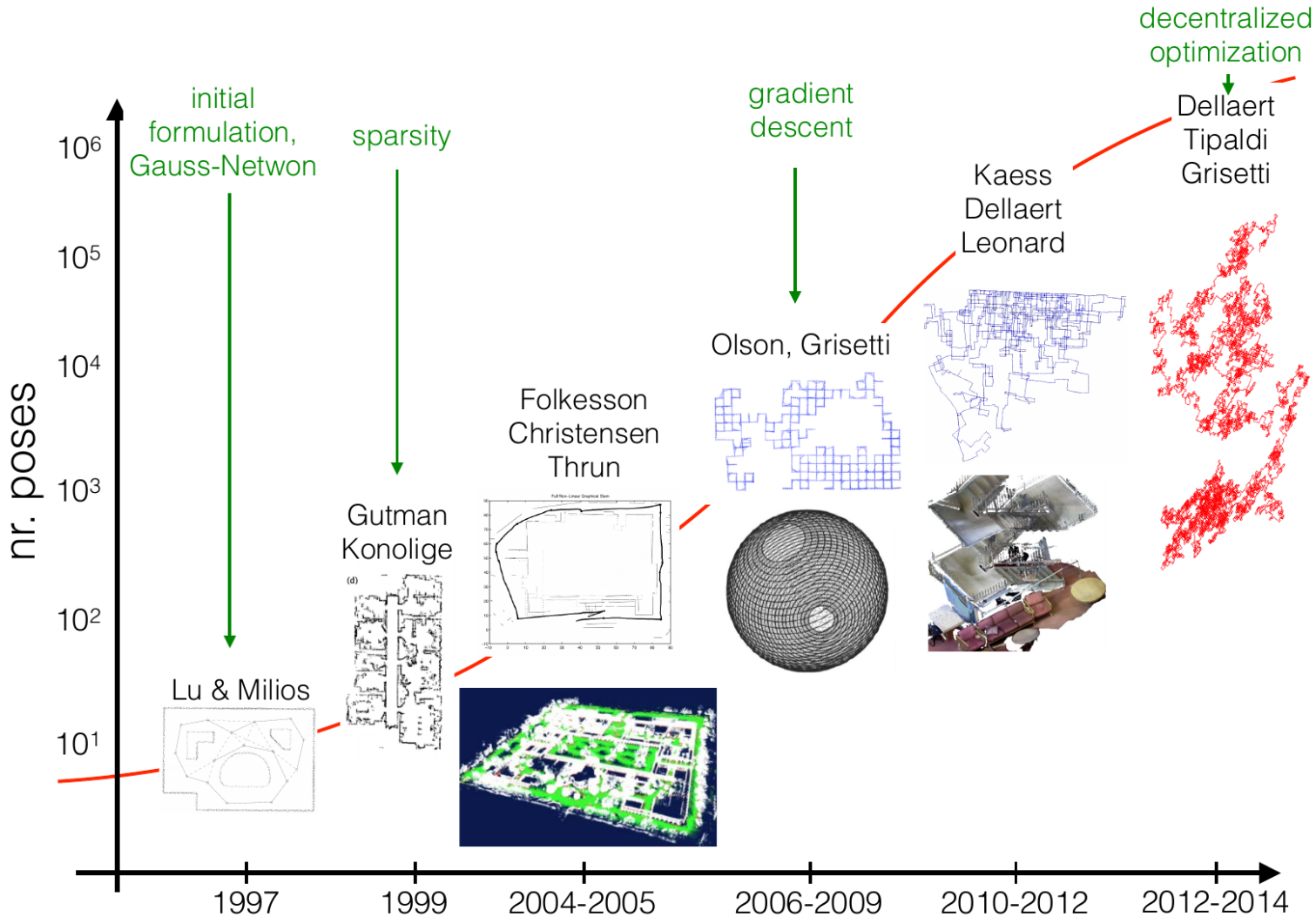

Figure courtesy of Luca Carlone

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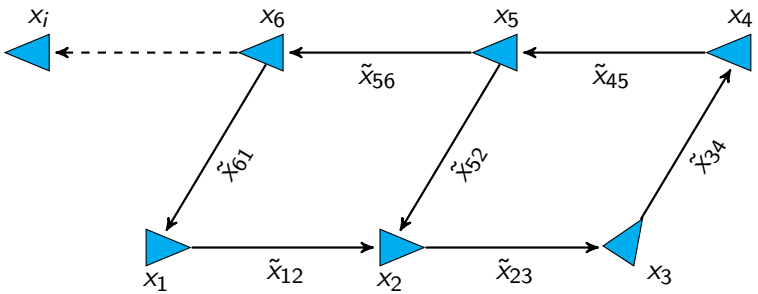

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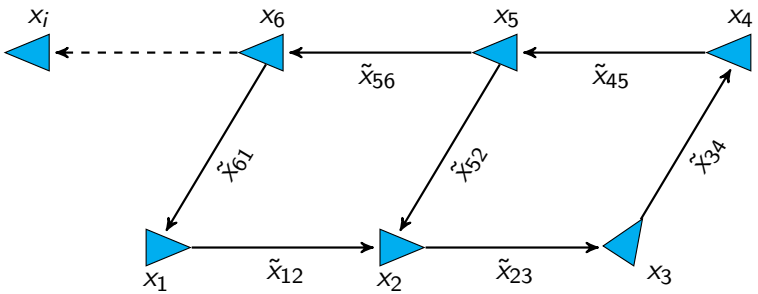

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But: How to handle nonconvexity?

Step 1: Reformulating pose-graph SLAM

Standard model assumes *exponentiated Gaussian* rotational noise:

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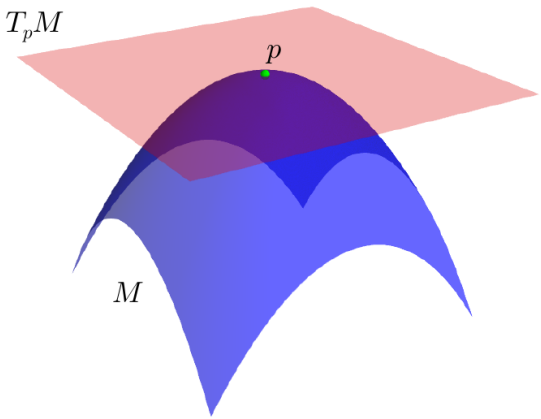

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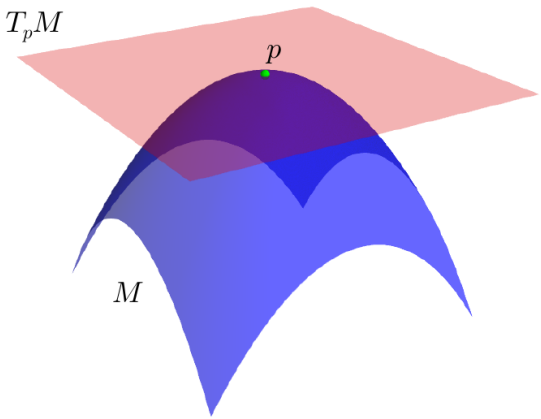

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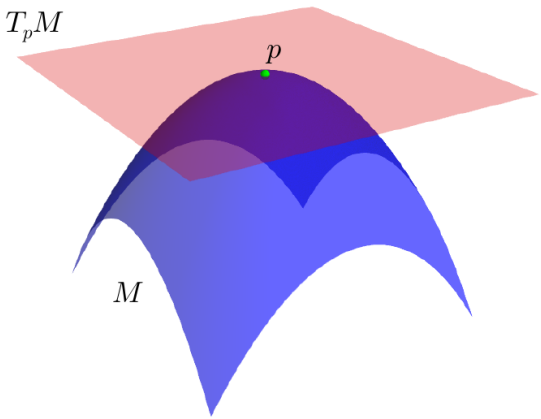

Main insight: This is not a "canonical" choice

$\Rightarrow$ Perhaps there is a more convenient formulation ...

Step 1: Reformulating pose-graph SLAM

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$$p(R; M, \kappa) = \frac{1}{c_d(\kappa)} \exp\left(\kappa \operatorname{tr}(M^{\mathsf{T}} R)\right), \quad R \in \mathrm{SO}(d)$$

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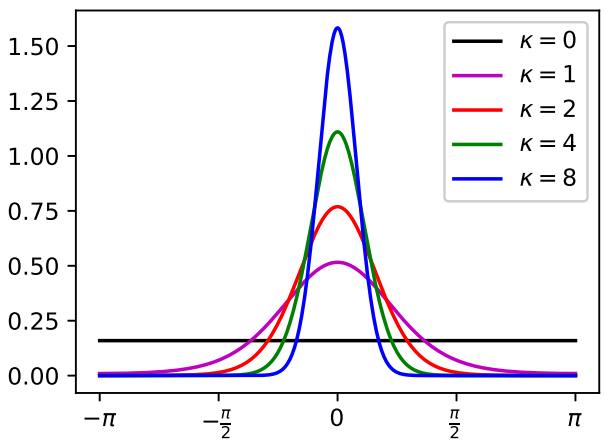

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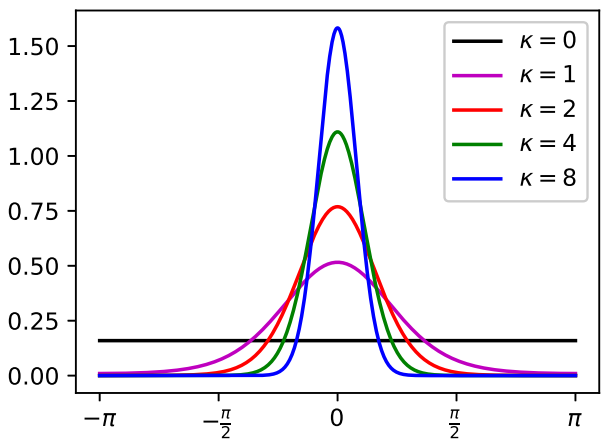

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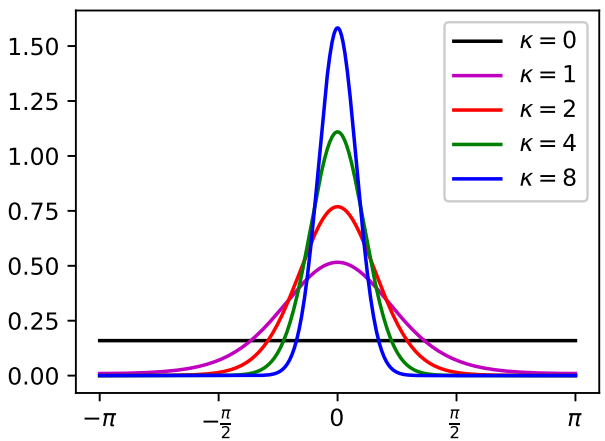

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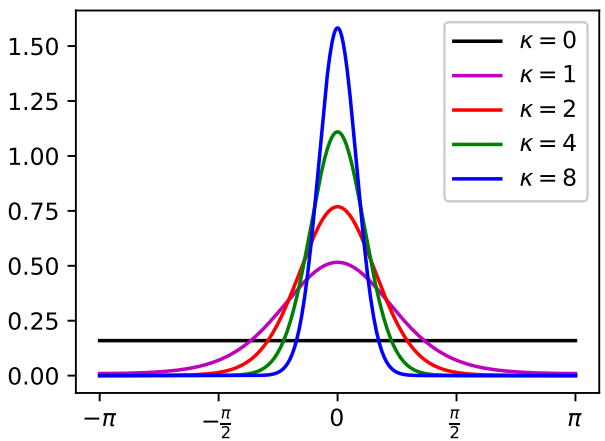

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- Exponential family $\Rightarrow$ MLE has simple algebraic form

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Payoff: Our formulation has a *convex quadratic* objective.

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$$p^*_{\mathrm{MLE}} = \min_{\substack{t_i \in \mathbb{R}^d \\ R_i \in \mathrm{SO}(d)}} \sum_{(i,j)\in\vec{\mathcal{E}}} \kappa_{ij} \left\| R_j - R_i \tilde{R}_{ij} \right\|_F^2 + \tau_{ij} \left\| t_j - t_i - R_i \tilde{t}_{ij} \right\|_2^2$$

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Solving for $t \triangleq (t_1, \ldots, t_n)$ in terms of $R \triangleq (R_1, \ldots, R_n)$ using a generalized Schur complement...

Step 2: Simplifying the maximum-likelihood estimation

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[Lots of algebra...]

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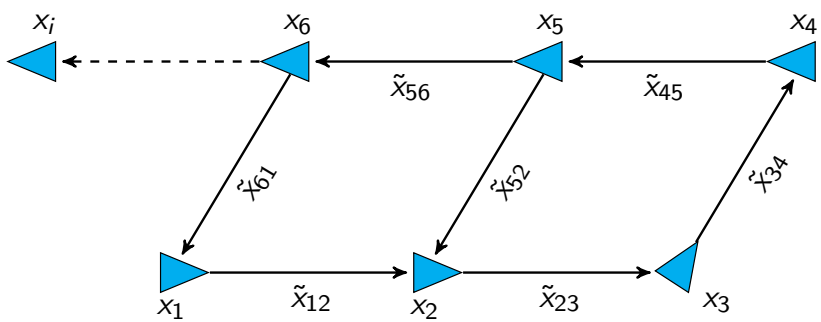
- Data matrices $L(\tilde{G}^\rho)$, $\tilde{T}$, Ω, Π have *simple interpretations* in G



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$\Rightarrow$ *Expanding MLE's feasible set to* $\mathcal{C}$ gives a **convex relaxation**

The Main Idea

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Proposition (Rosen, Carlone, et al. 2016)

Let $\underline{Q}$ be the matrix constructed using the true relative transforms $\underline{x}_{ij}$. There is a constant $\beta \triangleq \beta(\underline{Q}) > 0$ s.t., if $\|\tilde{Q} - \underline{Q}\|_2 < \beta$, then:

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$\Rightarrow$ We can recover an *exact* MLE by (numerically) solving SDP!

The direct approach

$$p^*_{\text{SDP}} = \min_{Z \in \mathbb{S}^{dn}_+} \operatorname{tr}(\tilde{Q}Z)$$

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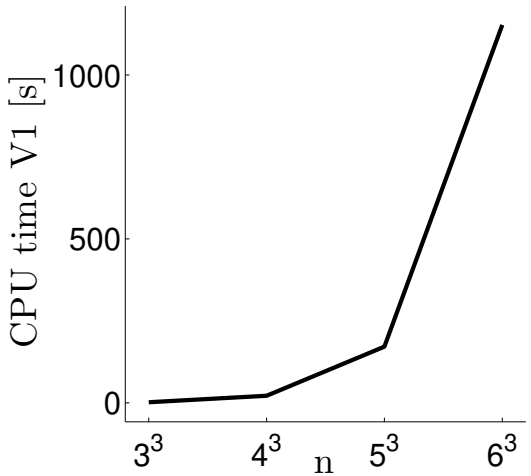

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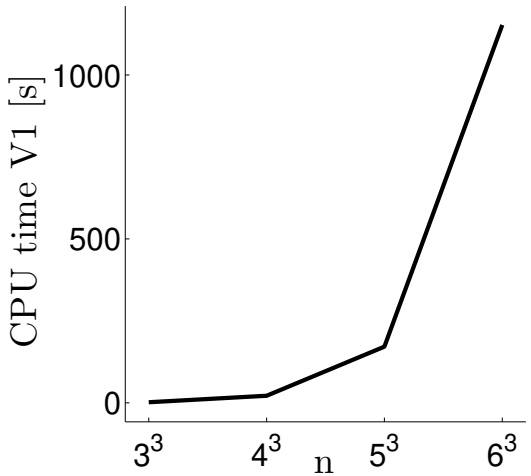

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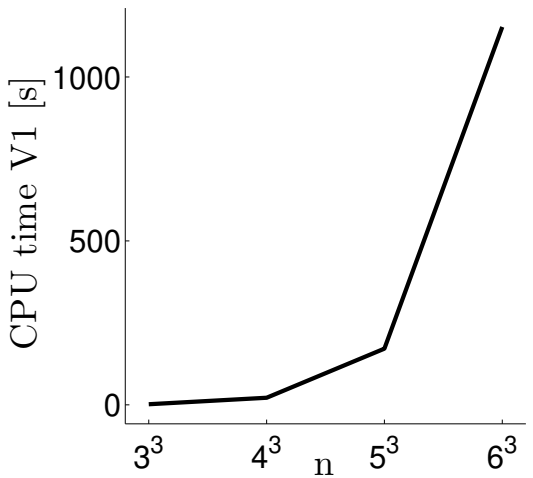

General SDP of order n:

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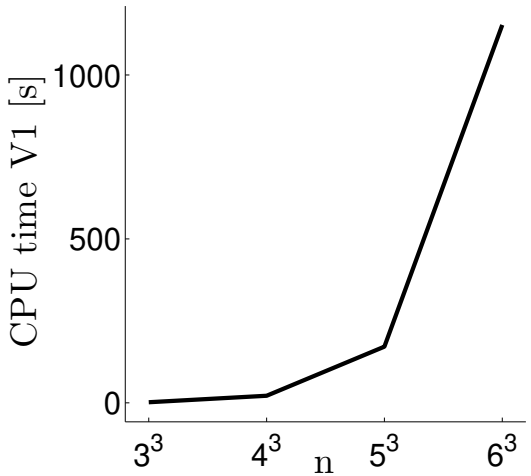

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$\Rightarrow$ Matrices have dimension $O(n^2)$

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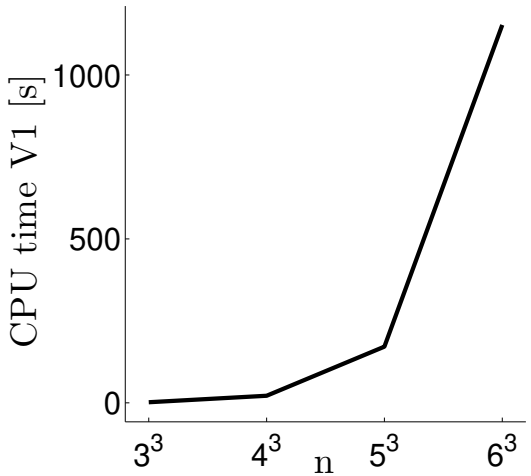

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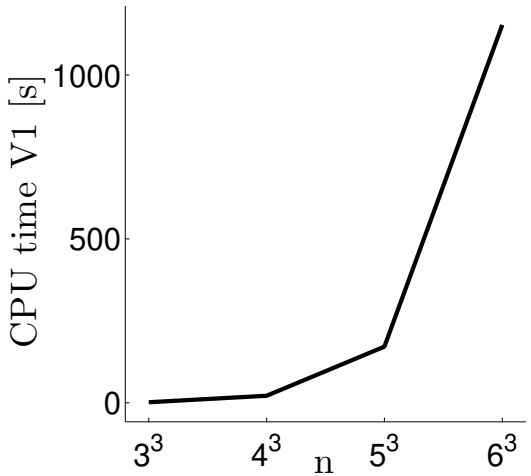

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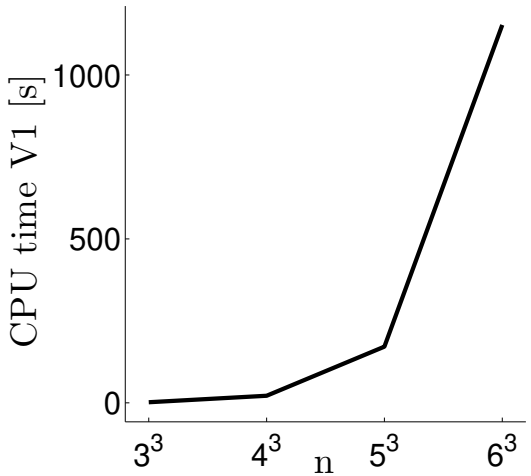

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But: Maybe we can build a *specialized* solver for PGO?

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Exploiting low-rank structure

We expect a *low-rank* solution $Z^* = {Y^*}^{\mathsf{T}} Y^*$ for:

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- $r \geq \mathrm{rank}(Z^*)$ for some optimal $Z^* \Rightarrow (Y^*)^{\mathsf{T}} Y^*$ solves SDP.

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Exploiting geometric structure[5]

Rank-restricted SDP, NLP form

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Notice: This is the definition of the *Stiefel manifold*:

$$\operatorname{St}(k, n) \triangleq \left\{ Y \in \mathbb{R}^{n \times k} \mid Y^{\mathsf{T}} Y = I_k \right\}.$$

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Payoff: This is an *unconstrained* optimization problem.

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Ensuring global optimality

Simplified pose-graph MLE

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$$p^*_{\text{SDPLR}} = \min_{Y \in \text{St}(d,r)^n} \text{tr}(\tilde{Q} Y^{\text{T}} Y)$$

Ensuring global optimality

Simplified pose-graph MLE

$$p^*_{\mathrm{MLE}} = \min_{R \in \mathrm{SO}(d)^n} \mathrm{tr}(\tilde{Q} R^{\mathsf{T}} R)$$

$$\Downarrow$$

Semidefinite relaxation

$$p^*_{\mathrm{SDP}} = \min_{Z \in \mathbb{S}^{dn}_+} \mathrm{tr}(\tilde{Q} Z)$$

$$\text{s.t.} \quad \mathrm{BlockDiag}_d(Z) = (I_d, \ldots, I_d)$$

$$\Downarrow$$

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Question:

What have we actually gained?

Ensuring global optimality

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Proposition (Boumal, Voroninski, and Bandeira 2016)

If $Y \in \text{St}(d, r)^n$ is a rank deficient 2nd-order critical point for rank-restricted NLP, then:

- *Y is a global minimizer of Riemannian NLP*
- *$Z^* = Y^{\text{T}} Y$ is a solution of the semidefinite relaxation.*

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- *Y is a global minimizer of Riemannian NLP*
- *$Z^* = Y^{\mathsf{T}} Y$ is a solution of the semidefinite relaxation.*

$\Rightarrow$ We can use (fast) *local* search to find *globally optimal* solutions!

The Riemannian Staircase

Semidefinite relaxation

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$$\text{s.t.} \quad \text{BlockDiag}_d(Z) = (I_d, \ldots, I_d)$$

Riemannian rank-restricted NLP

$$p^*_{\text{SDPLR}} = \min_{Y \in \text{St}(d,r)^n} \operatorname{tr}(\tilde{Q}Y^{\mathsf{T}}Y)$$

Input: Initial point $Y \in \text{St}(d, r_0)^n$, $r_0 \geq d$.
Output: Symmetric factor Y^* of SDP solution $Z^* = {Y^*}^{\mathsf{T}} Y^*$.

1: **for** $r = r_0, \ldots, dn+1$ **do**
2: $Y^* \leftarrow \text{SecondOrderLocalSearch}(Y)$ (Riemannian NLP).
3: **if** $\text{rank}(Y^*) < r$ **then**
4: **return** Y^*
5: **else**
6: Set $Y \leftarrow \begin{pmatrix} Y^* \\ 0_{1 \times dn} \end{pmatrix}$.
7: **end if**
8: **end for**

The Riemannian Staircase

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SE-Sync

Simplified pose-graph MLE

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$\Downarrow$

SDP relaxation

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The SE-Sync algorithm:

1. Find low-rank factor Y^* using fast (2nd-order) NLP method in Riemannian Staircase.

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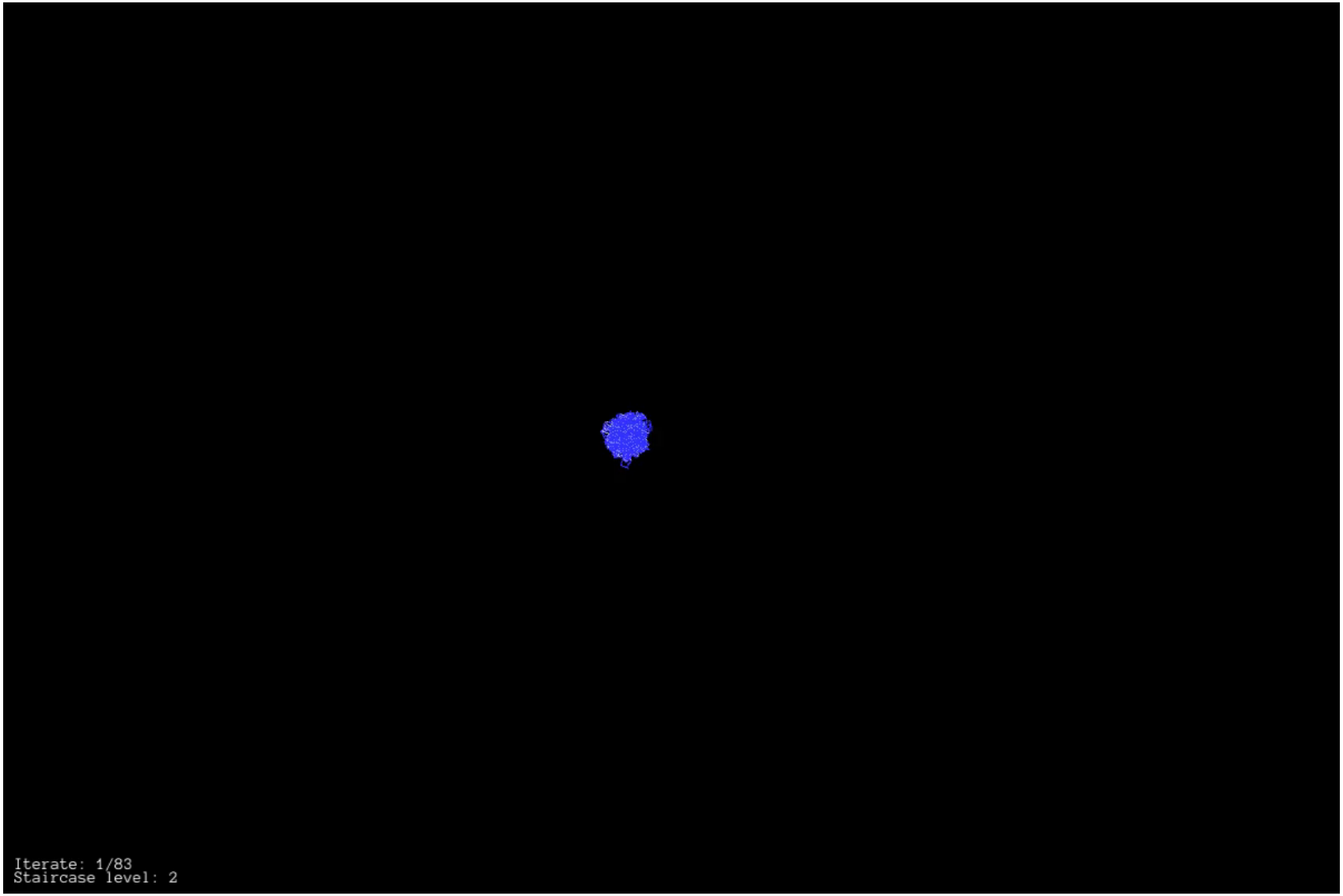
3. Round $Y^* \to \hat{R} \in \text{SO}(d)^n$ using truncated SVD.
4. Return $\{\hat{R}, p^*_{\text{SDP}}\}$.

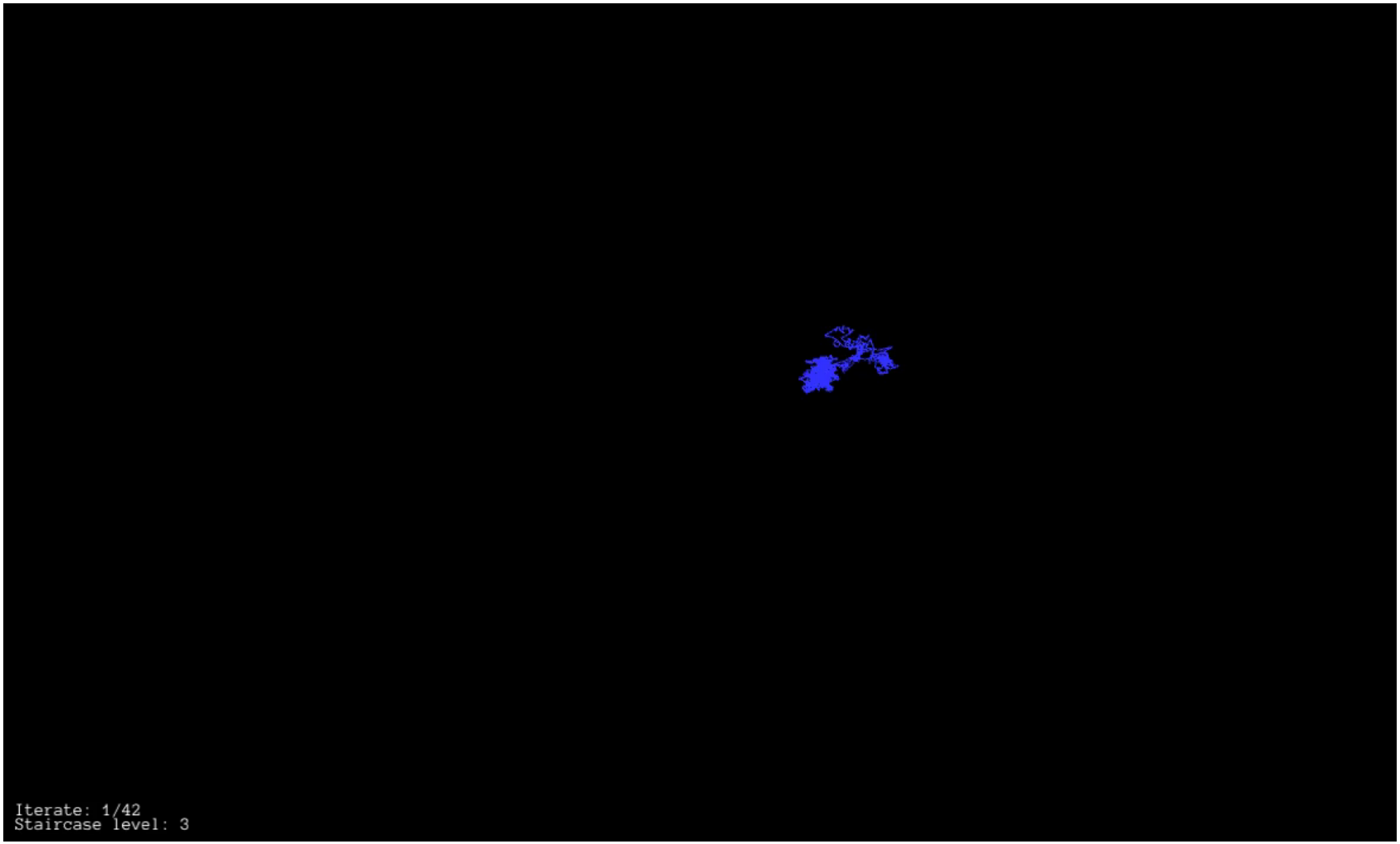
Payoff: If exactness holds:

- $\hat{R}$ is *globally optimal*
- $\text{tr}(\tilde{Q} \hat{R}^{\text{T}} \hat{R}) = p^*_{\text{SDP}}$ *certifies* it

Visualization I: Grid world

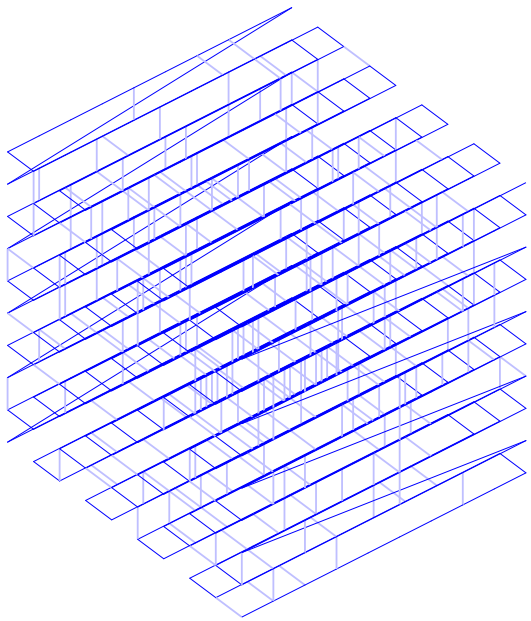


Visualization II: Parking garage



Experimental results I: Grid world simulations

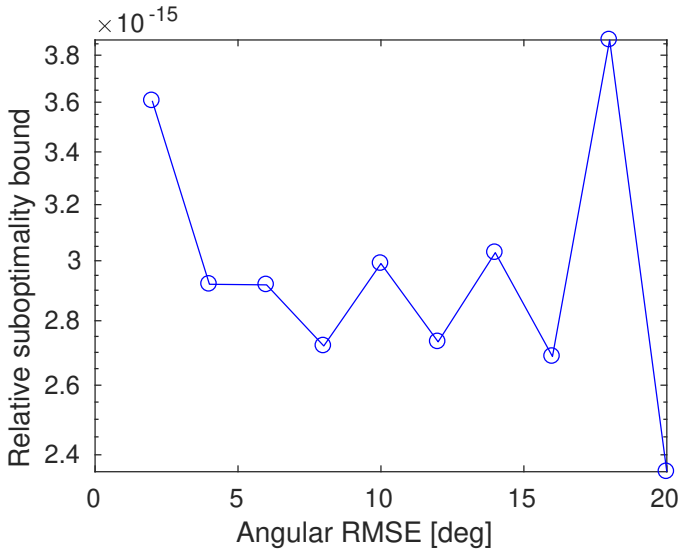
Question: How do noise and problem size affect performance?
Test: Simulate random grid-world, varying κ, τ, and n:

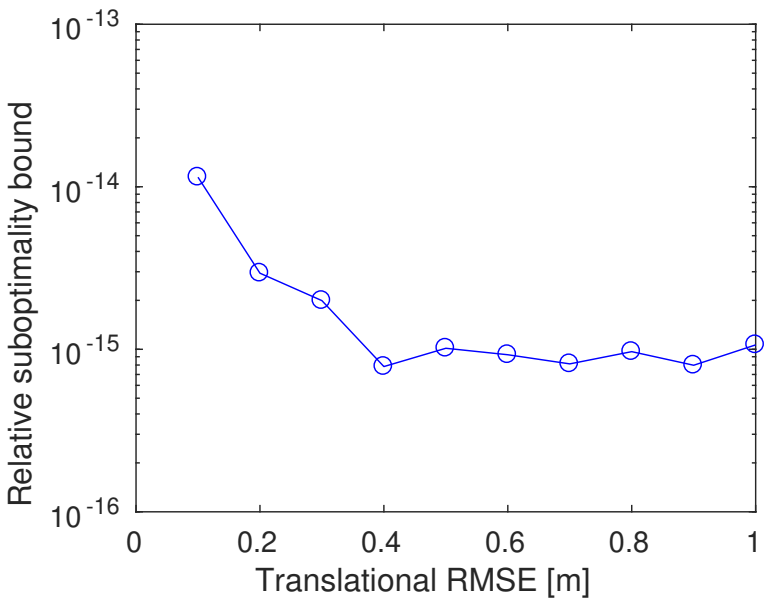


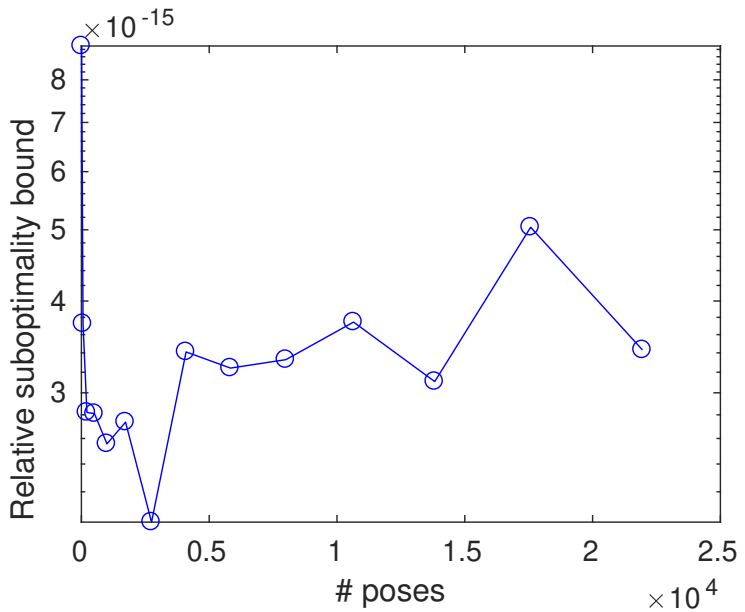
$$\tilde{t}_{ij} = R_i^T(t_j - t_i) + \delta t_{ij}, \qquad \delta t_{ij} \sim \mathcal{N}(0, \tau^{-1} I_d)$$
$$\tilde{R}_{ij} = R_i^T R_j \cdot \delta R_{ij}, \qquad \delta R_{ij} \sim \text{Langevin}(I_d, \kappa)$$

Baseline: Gauss-Newton using *chordal initialization*

Experimental results I: Grid world simulations







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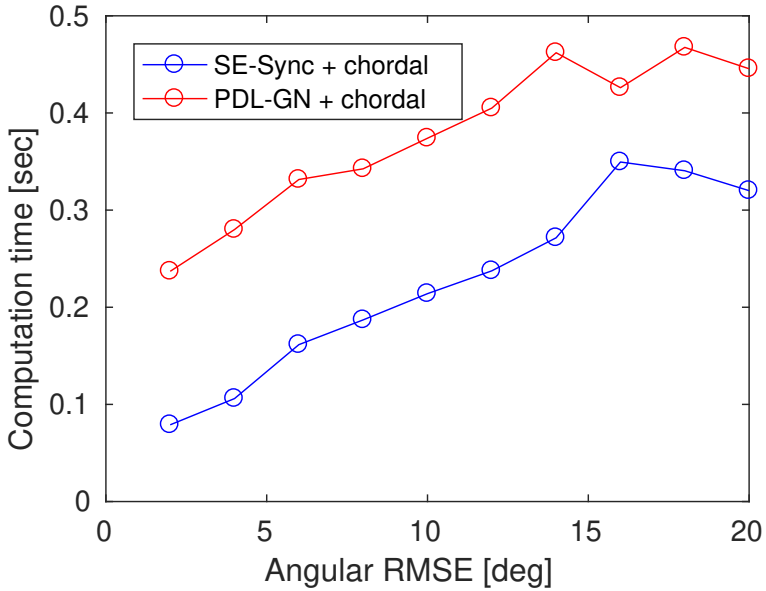

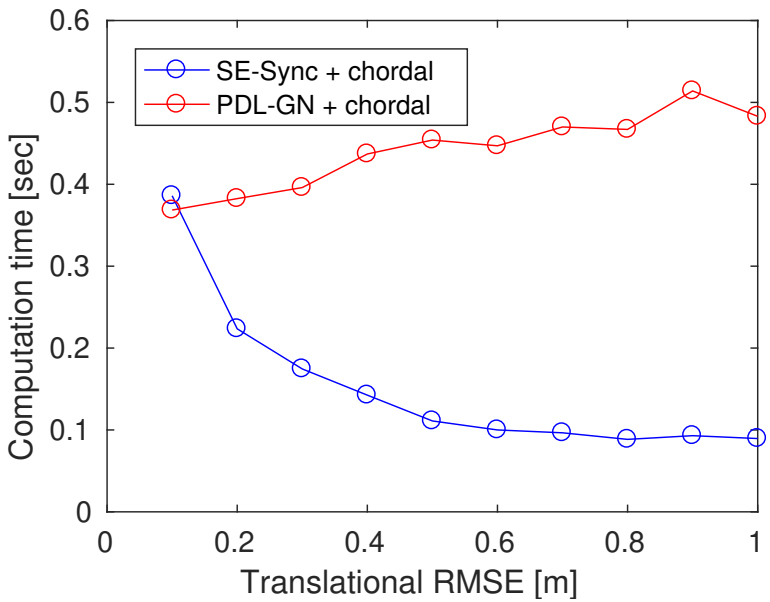

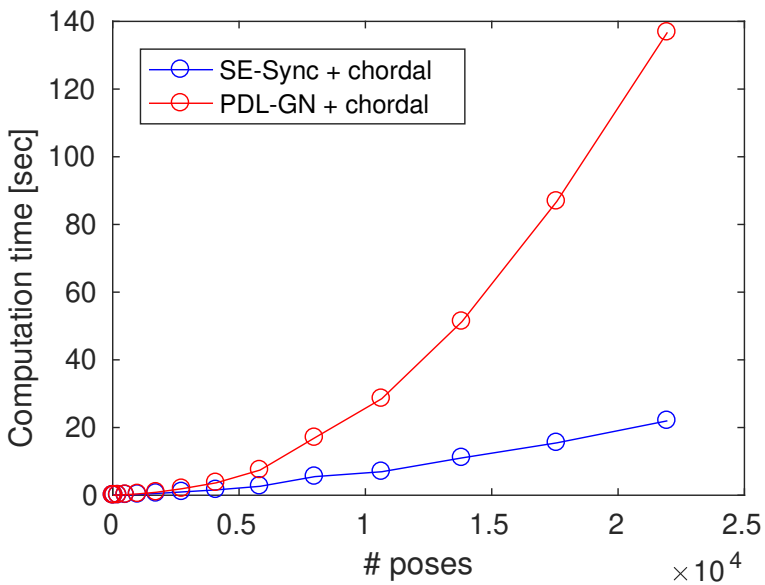

Experimental results II: Large-scale SLAM benchmarks

	# Poses	PDL-GN (GTSAM)		SE-Sync		
		Objective value	Time [s]	Objective value	Time [s]	Rel. suboptimality
csail	1045	3.170×10^1	0.029	3.170×10^1	0.010	7.844×10^{-16}
intel	1728	5.235×10^1	0.120	5.235×10^1	0.071	1.357×10^{-16}
ais2klinik	15115	1.885×10^2	12.472	1.885×10^2	1.981	2.412×10^{-15}
garage	1661	1.263×10^0	0.415	1.263×10^0	0.468	1.618×10^{-14}
cubicle	5750	7.171×10^2	2.456	7.171×10^2	0.754	2.061×10^{-15}
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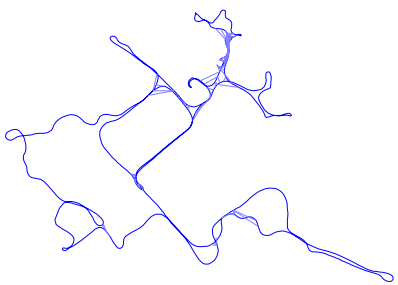
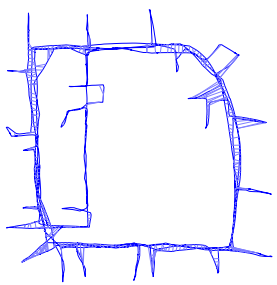
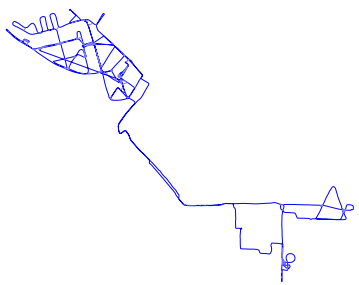


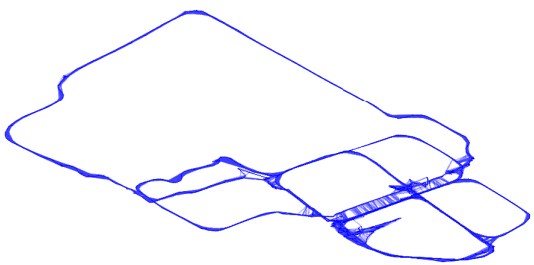
csail



intel



ais2klinik



rim

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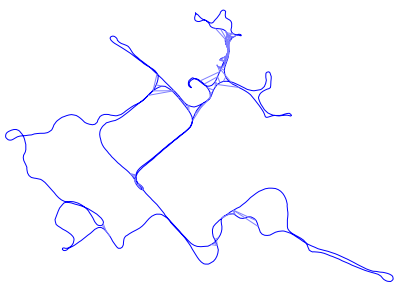

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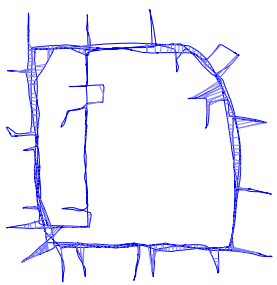

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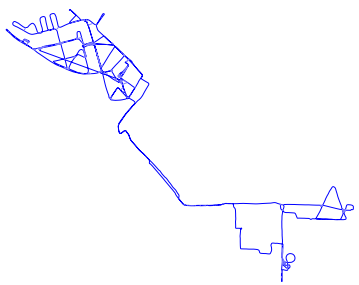

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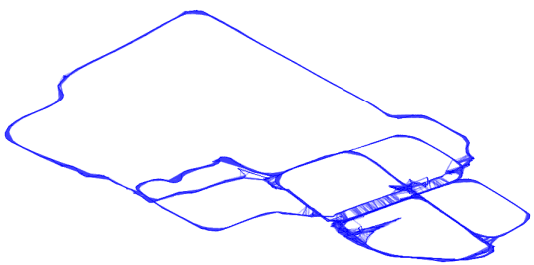

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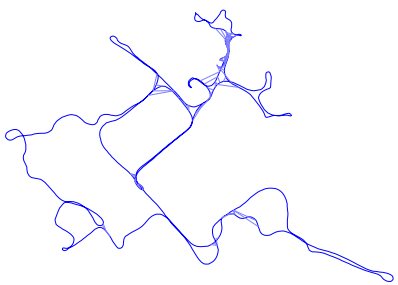

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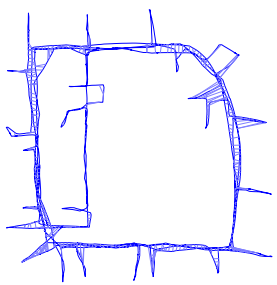

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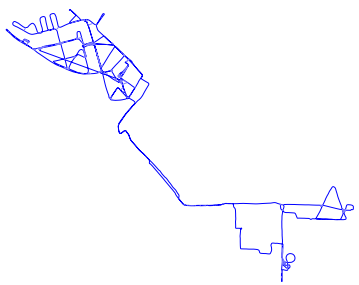

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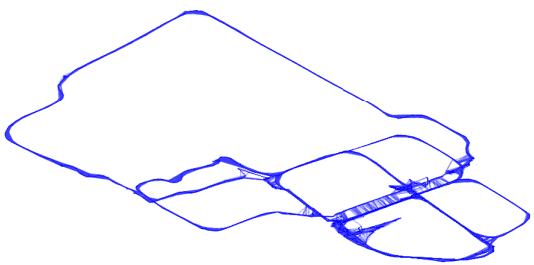

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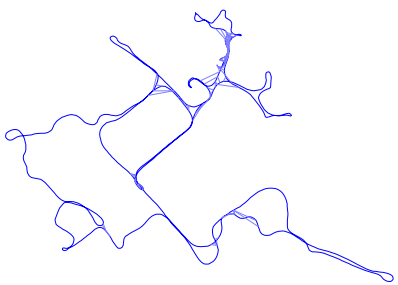

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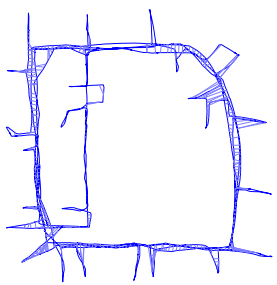

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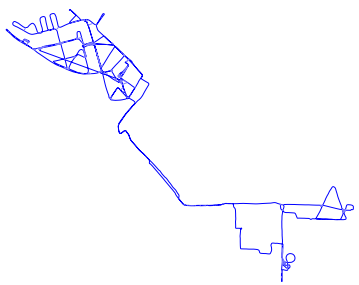

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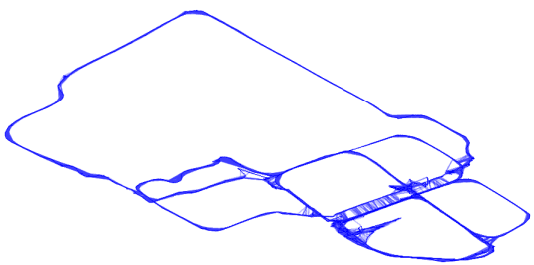

rim

Contributions

SE-Sync: A *certifiably correct* algorithm for pose-graph SLAM[6]

- Recovers *globally optimal* estimates for moderate noise (up to 10x typical levels)
- Significantly faster than prior state-of-the-art techniques

[6] D.M. Rosen, L. Carlone, et al. "SE-Sync: A Certifiably Correct Algorithm for Synchronization over the Special Euclidean Group". In: *Intl. J. of Robotics Research* 38.2–3 (Mar. 2019), pp. 95–125

[7] D.M. Rosen. "Towards Provably Robust Machine Perception". Presented at Robotics: Science and Systems (RSS) in the workshop "RSS Pioneers". June 2019

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SE-Sync: A *certifiably correct* algorithm for pose-graph SLAM[6]

- Recovers *globally optimal* estimates for moderate noise (up to 10x typical levels)
- Significantly faster than prior state-of-the-art techniques

$\Rightarrow$ First practical alg. *provably* able to recover correct solutions

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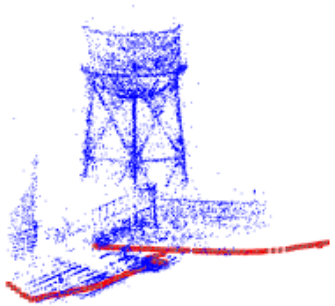
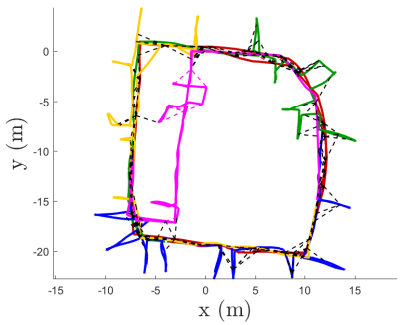
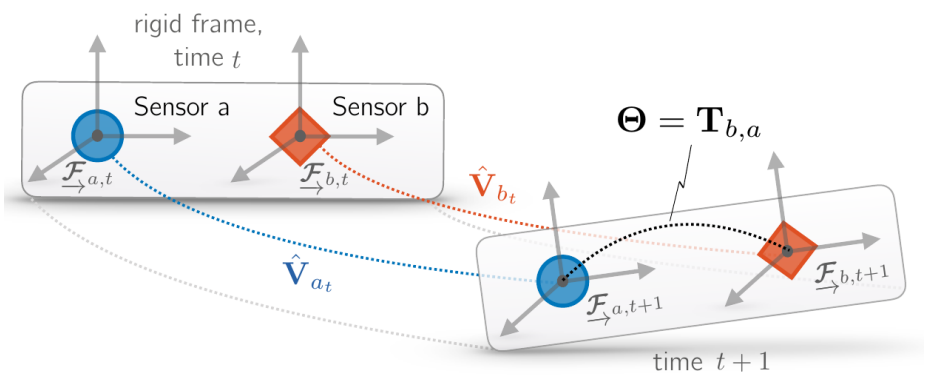
Applications of certifiable estimation

- Rotation averaging[1]
- Two-view registration[2]
- Sensor calibration[3]
- Image segmentation[4]
- Distributed SLAM[5]







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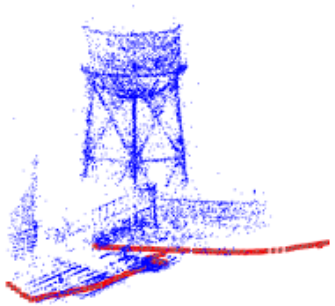

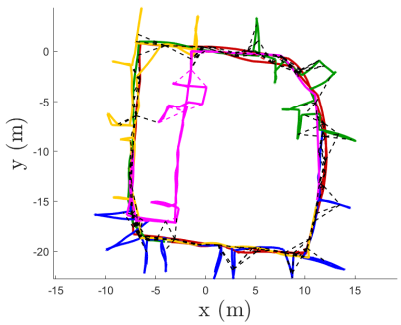

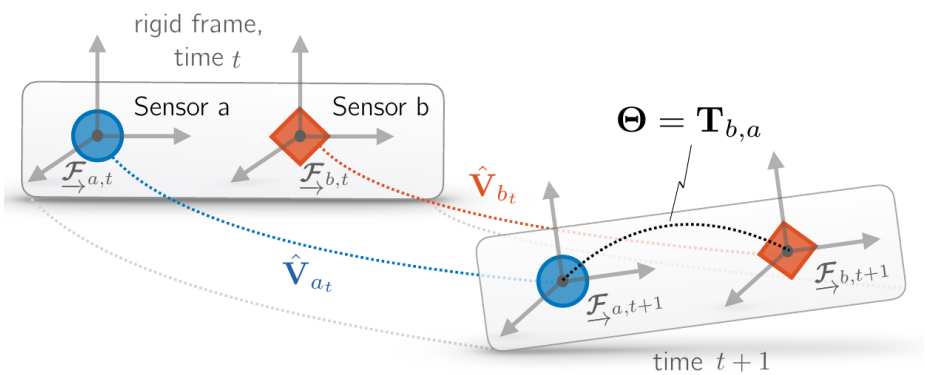

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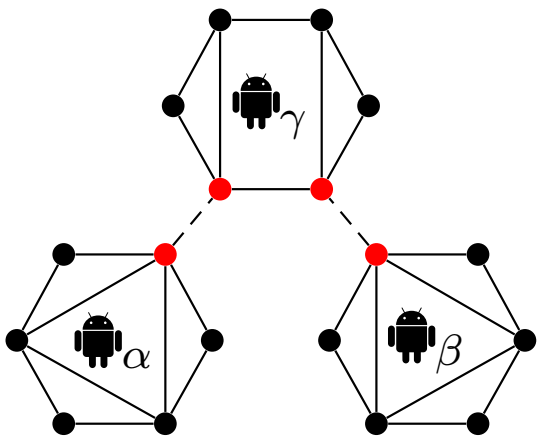

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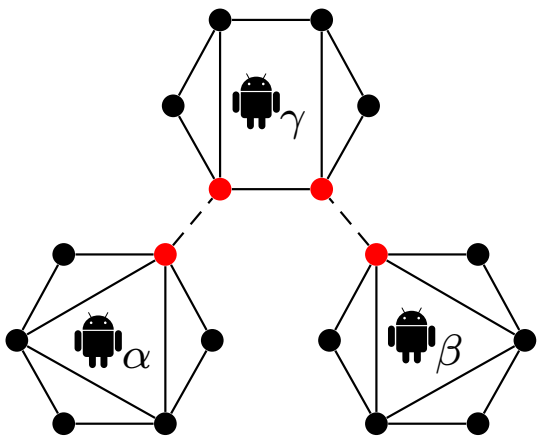

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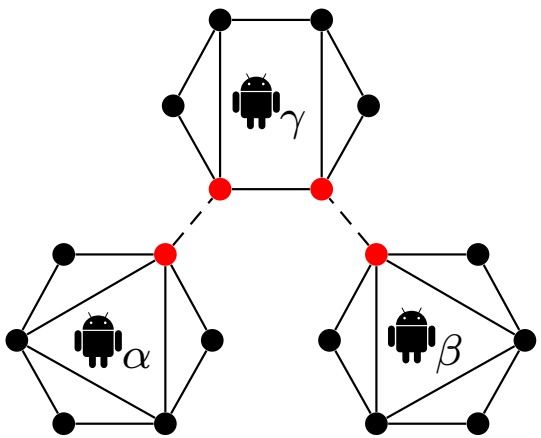

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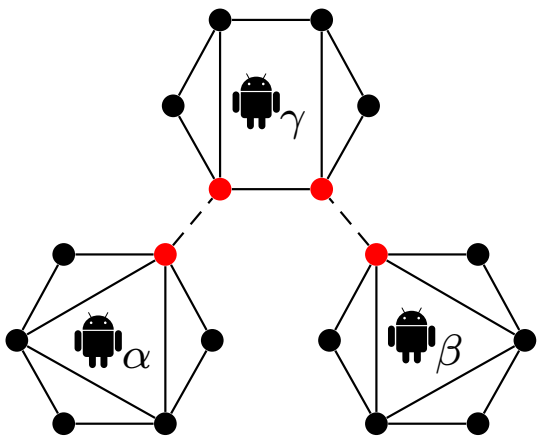

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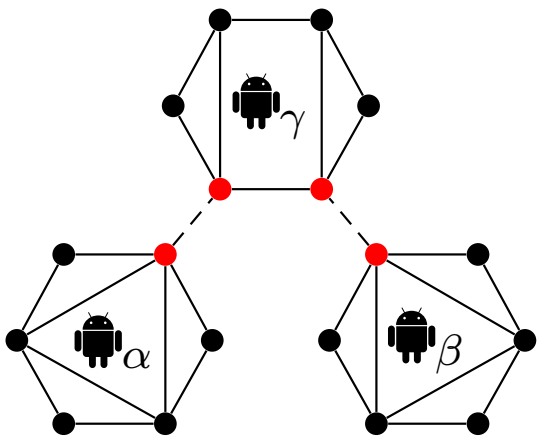

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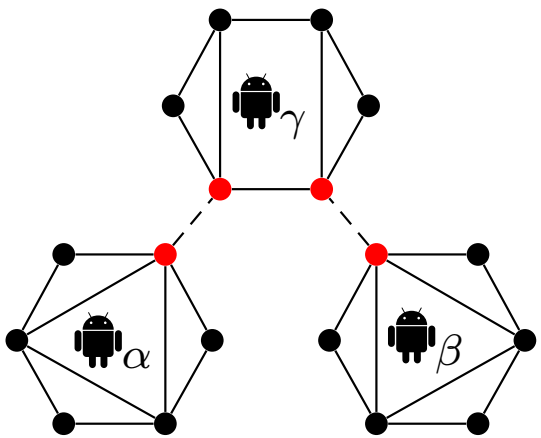

$\Rightarrow$ *Fully decentralized* and *privacy preserving*

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Application: Distributed metric-semantic SLAM[1]

Kimera-Multi: a System for Distributed Multi-Robot Metric-Semantic Simultaneous Localization and Mapping

Yun Chang, Yulun Tian, Jonathan P. How, Luca Carlone

Massachusetts Institute of Technology

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Future directions: General certifiable estimation[10]

- General and robust extensions via moment relaxation
- Scalable semidefinite optimizers

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